1 2.3 Linear functions

Definition 1 \( f(x) = mx + b, m \neq 0 \) is a linear function of \( x \) and this is a nonvertical linear function.

In the figure, the lines intersect at the same point but differs in the steepness (the slope).

Definition 2 The slope \( m = \frac{y_2 - y_1}{x_2 - x_1} \), \( x_1 \neq x_2 \) where the line passes through the points \( P_1(x_1, y_1) \) and \( P_2(x_2, y_2) \).

The numerator \( y_2 - y_1 \) is the vertical rise from \( P_1 \) to \( P_2 \) and the denominator \( x_2 - x_1 \) is the horizontal run from \( P_1 \) to \( P_2 \).

Example 3 Graph the line with slope \( \frac{3}{2} \) and passes through the point \((1, 2)\).

Lines that have a positive slope slant upward from left to right and lines that have a negative slope slant downward from left to right.

Example 4 Find the slope of the line passing through the points whose coordinates are: 1) \((3, 4)\) and \((5, 10)\). \((\text{ans: } m=3)\) 2) \((-4, -3)\) and \((1, -2)\). \((\text{ans: } m=\frac{1}{5})\)

Special cases:
1-Vertical line passes through the point \((a, 0)\). Slope: \( \text{Equation:} \)
Is it a function or not?
2-Horizontal line passes through the point \((0, b)\). Slope: \( \text{Equation:} \)
Is it a function or not?

Slope-Intercept Form
The graph of \( f(x) = mx + b \) is a line with slope \( m \) and y-intercept \((0, b)\).

Example 5 Graph the following lines 1) \( f(x) = 2x - 1 \) 2) \( f(x) = -3x + 4 \)

Example 6 Find the equation of a line with y-intercept \((0, 3)\) and slope \( \frac{3}{4} \).

Point-slope Form
The graph of \( y - y_1 = m(x - x_1) \) is a line that has slope \( m \) and passes through the point \((x_1, y_1)\).

Example 7 Find the equation of a line with slope \( \frac{2}{3} \) and passes through the point \((-3, 4)\).

Example 8 Find the equation of a line that passes through the points \((3, 1)\) and \((-1, 4)\).

Example 9 Find the equation of a line with x-intercept \((3, 0)\) and passes through the point \((2, 1)\).

General form of the equation of a line: \( Ax + By + C = 0 \), \( A, B \neq 0 \).
Example 10 Write the equation of the line in its general form. \[ y = -3x + 1. \]

Example 11 Determine the slope, \(y\)-intercept and \(x\)-intercept of the following lines: 1) \(f(x) = -\frac{1}{2}x + 3\). 2) \(3x + 2y = 6\).

Example 12 Find the value of \(x\) in the domain of \(f(x) = 5x + 12\), for which \(f(x) = 2\).

Theorem 13 Real solutions and \(x\)-intercept theorem: For every function \(f\), the real number \(c\) is a solution of \(f(x) = 0\) if and only if \((c, 0)\) is an \(x\)-intercept of the graph of \(y = f(x)\).

Example 14 Let \(f(x) = 2x + 4\). Find the real solution of \(f(x) = 0\) and then graph \(y = f(x)\). Compare the solution of \(f(x) = 0\) with the \(x\)-intercept of the graph of \(f\).

Example 15 Let \(f_1(x) = \frac{3}{2}x + 6\) and \(f_2(x) = -2x + 1\). Find the value of \(x\) for which \(f_1(x) = f_2(x)\).

Parallel and perpendicular lines: \(l_1: f_1(x) = m_1x + b_1 \& l_2: f_2(x) = m_2x + b_2\)

Definition 16 \(l_1\) and \(l_2\) are parallel if \(m_1 = m_2\).

Definition 17 \(l_1\) and \(l_2\) are perpendicular if \(m_1m_2 = -1\).

Example 18 Find the equation of a line that passes through \((1,3)\) and parallel to the line \(3x + 4y = -24\).

Example 19 Find the equation of a line that passes through \((1,2)\) and perpendicular to the line \(2x - y = 7\).

Example 20 Find a point \(P(x, y)\) on the graph of the equation \(y = x^2\) such that the slope of the line through the point \((3, 9)\) and \(P\) is \(\frac{5}{12}\).

Example 21 Let \(f\) be a linear function such that \(f(9) = 0\) and the graph of \(f\) is parallel to the line \(x - 3y - 4 = 0\), then find \(f(3)\). (ans: -2)

Example 22 Find the value of \(k\) where the line through the points \((4, -1)\) and \((k, 2)\) is perpendicular to the line \(2x - 5y = 1\). (ans: \(\frac{13}{5}\))

Example 23 Find the \(x\)-intercept and the \(y\)-intercept of the line passing through the points \((-2, -1)\) and \((1, 3)\). (ans: \(\frac{-5}{4}, \frac{5}{4}\))

Example 24 If \(f(5) = -2\), \(f(1) = 0\), and \(f\) is a linear function, then find \(f(-4)\). (ans: \(\frac{-2}{5}\))

Example 25 If the points \((k, -8), (-1, 7)\) and \((2, -2)\) lie on the same straight line, then find the value of \(k\).

Example 26 Let \(f\) be a linear function such that \(f(9) = 0\) and the graph of \(f\) is parallel to the line \(x - 3y - 4 = 0\), then find the value of \(f(3)\) (ans: -2)

Example 27 Find the \(y\)-intercept of the line passing through \((2, -5)\) and perpendicular to the line \(3x + 2y = 5\). (ans: \(-\frac{19}{3}\))