1 Section 10.3 The Inverse of a Matrix

Multiplicative Inverse of a Matrix

If \( A \) is a square matrix of order \( n \), then the inverse of matrix \( A \), denoted by \( A^{-1} \), has the property that \( A \cdot A^{-1} = A^{-1} \cdot A = I_n \) where \( I_n \) is the identity matrix of order \( n \).

Note that not all square matrices has a multiplicative inverse.

A procedure for finding the inverse uses elementary row operations. To the matrix \( A \) we will merge the identity matrix \( I \) to the right of \( A \) and denote this new matrix by \([A : I]\). Now, we will use elementary row operations to produce \([I : A^{-1}]\).

Example 1 Find \( A^{-1} \) if \( A = \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix} \).

Example 2 Find the inverse of the matrix \( A = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 3 & 6 \\ -1 & -1 & 2 \end{bmatrix} \).

A singular matrix is a matrix that does not have a multiplicative inverse. A matrix that has a multiplicative inverse is a non singular matrix.

If there are all zeros in a row of the original matrix (after we apply the elementary row operations), then the matrix does not have an inverse.

Example 3 Show that the matrix \( \begin{bmatrix} 1 & -6 & 4 \\ 3 & 4 & 2 \\ 5 & 3 & 5 \end{bmatrix} \) is a singular matrix.

Systems of equations can be solved by finding the inverse of the coefficient matrix in the following steps:
1) Write the linear system as a matrix equation in the form \( AX = B \).
2) The solution is \( X = A^{-1}B \).

Example 4 Find the solution set of the following system of equations by using the inverse of the coefficient matrix:
1) \( \begin{align*}
3x_1 + 4x_2 &= -1 \\
3x_1 + 5x_2 &= 1
\end{align*} \)
2) \( \begin{align*}
x_1 + 7x_3 &= 20 \\
2x_1 + x_2 - x_3 &= -3 \\
7x_1 + 3x_2 + x_3 &= 2
\end{align*} \)

Example 5 True or False: 1) If \( AB = O \), then \( A = O \) or \( B = O \). (\( A = \begin{bmatrix} 2 & -3 \\ -6 & 9 \end{bmatrix} \) and \( B = \begin{bmatrix} -3 & 15 \\ -2 & 10 \end{bmatrix} \))
2) If \( A \) has an inverse and \( AB = O \), then \( B = O \).
3) If \( AB = AC \), then \( B = C \). (\( A = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix} \), \( B = \begin{bmatrix} 3 & 4 \\ 1 & 5 \end{bmatrix} \), and \( C = \begin{bmatrix} 4 & 7 \\ 3 & 11 \end{bmatrix} \))
4) If \( A \) has an inverse and \( AB = AC \), then \( B = C \).
Example 6 Show that if \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) and \( ad - bc \neq 0 \), then \( A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \).

Example 7 Find the inverse (if possible) of 1) \( \begin{bmatrix} 5 & 6 \\ 3 & 4 \end{bmatrix} \) 2) \( \begin{bmatrix} 5 & 1 \\ 10 & 2 \end{bmatrix} \).

Example 8 Show that \((AB)^{-1} = B^{-1}A^{-1}\).