1. Consider the IVP 
\[ y' = (t + y - 1)^2, \quad y(0) = 2 \]
(a) Solve the IVP in terms of elementary functions. [Set \( u = t + y - 1 \).]
(b) Use Euler’s method with \( h = 0.1 \) and \( h = 0.05 \) to obtain approximate values of the solution of the IVP at \( t = 0.5 \). Compare the approximate values with the exact values using part (a).

Use Euler’s method to obtain approximations to the following IVPs to the indicated value:

2. \( y' = 2t - 3y, \ y(1) = 5, \ y(1.5) \) with \( h = 0.05 \).
3. \( y' = 4t - 2y, \ y(0) = 2, \ y(0.5) \) with \( h = 0.05 \).
4. \( y' = 1 + y^2, \ y(0) = 0, \ y(1) \) with \( h = 0.1 \).
5. \( y' = 2t - 3y, \ y(1) = 5, \ y(1.5) \) with \( h = 0.05 \).
6. \( y' = t^2 + y^2, \ y(0) = 1, \ y(0.5) \) with \( h = 0.05 \).
7. \( y' = e^{-y}, \ y(0) = 0, \ y(1) \) with \( h = 0.1 \).
8. \( y' = t + y^2, \ y(0) = 0, \ y(0.5) \) with \( h = 0.05 \).
9. \( y' = (t + y)^2, \ y(0) = 0.5, \ y(0.5) \) with \( h = 0.05 \).
10. \( y' = ty + \sqrt{t}, \ y(0) = 1, \ y(1) \) with \( h = 0.1 \).
11. \( y' = ty^2 - \frac{y}{2}, \ y(1) = 1, \ y(1.5) \) with \( h = 0.05 \).
12. \( y' = y - y^2, \ y(0) = 0.5, \ y(1) \) with \( h = 0.1 \).


15. Let \( y(t) \) be the solution of the IVP
\[ y' = t^2 + y^2, \quad y(1) = 1 \]
(a) Use an ODE solver to obtain a graph of the solution on the interval \([1, 1.4]\).
(b) Using the step size \( h = 0.1 \), compare the results obtained from Euler’s method with the results from the modified Euler method (RK2) in the approximation of \( y(1.4) \).

16. Consider the IVP \( y' = 2y, \ y(0) = 1 \). The analytic solution is \( y(t) = e^{2t} \).
(a) Approximate \( y(0.1) \) using one step and Euler’s method.
(b) Find an error bound for the local truncation error in \( y_1 \).
(c) Compare the actual error in $y_1$ with your error bound.
(d) Approximate $y(0.1)$ using two steps and Euler’s method.
(e) Verify that the global truncation error for Euler’s method is $O(h)$ by comparing the errors in parts (a) and (d).

17. Solve the preceding exercise using the Runge-Kutta method of order 2. Its global truncation error is $O(h^2)$.

18. Repeat exercise 16 using the IVP $y' = -2y + t$, $y(0) = 1$. The analytic solution is $y(t) = \frac{1}{2}t - \frac{1}{4} + \frac{5}{4}e^{-2t}$.

19. If air resistance is proportional to the square of the instantaneous velocity, then the velocity $v$ of the mass $m$ dropped from a height $h$ is determined from

$$m \frac{dv}{dt} = mg - kv^2, \quad k > 0$$

Let $v(0) = 0$, $k = 0.125$, $m = 5$ slugs, and $g = 32$ ft/s$^2$.

(a) Use the Runge-Kutta method of order 4 with $h = 1$ to find an approximation of the velocity of the falling mass at $t = 5$ s.

20. Consider the IVP $y' = -y + 10 \sin 3t$ with $y(0) = 0$.

(a) Use the Runge-Kutta method of order 4 with $h = 0.1$ to approximate the solution in the interval $[0, 2]$.

(b) Using the result in (a) obtain an interpolating function and graph it. Find the positive roots of the interpolating function on the interval $[0, 2]$.

21. Show that when Euler’s method is used to approximate the solution of the IVP

$$y' = 5y, \quad y(0) = 1$$

at $t = 1$, then the approximation with step size $h$ is $(1 + 5h)^{1/h}$.

22. Show that when the Runge-Kutta method of order 2 is used to approximate the solution of the IVP

$$y' = y, \quad y(0) = 1$$

at $t = 1$, then the approximation with step size $h$ is $(1 + h + \frac{h^2}{2})^{1/h}$. The exact value of the solution of the IVP at $t = 1$ is $e$. Prove that the error $e - (1 + h + \frac{h^2}{2})^{1/h}$ approaches zero as $h \to 0$. Use the L’Hopital’s rule to show that

$$\lim_{h \to 0} \frac{\text{error}}{h^2} = \frac{e}{6} \approx 0.45305.$$ Derive the difference equation corresponding to Taylor’s method of order two for the following IVP:
23. \( y' = ty - y^2, \quad y(0) = -1. \)

24. \( y' = t - y, \quad y(0) = 0. \)

25. \( y' = t^2 + y, \quad y(0) = 0. \)

26. \( y' = t + 1 - y, \quad y(0) = 1. \)

27. Use Taylor’s method of order 2 with \( h = 0.25 \) to approximate the solution to the IVP

\[ y' = t + 1 - y, \quad y(0) = 1 \]

at \( t = 1 \). Compare these approximations to the actual solution \( y = t + e^{-t} \) evaluated at \( t = 1 \).

28. Use the Runge-Kutta method of order 4 to approximate the solution of the IVP

\[ y'' = t^2 + y^2, \quad y(0) = 1, \quad y'(0) = 0 \]

at \( t = 1 \).

29. Use the Runge-Kutta method of order 4 to approximate the solution of the IVP

\[ x' = 2x - y, \quad x(0) = 0, \]
\[ y' = 3x + 6y, \quad y(0) = -2. \]

at \( t = 1 \). Compare this approximation to the actual solution \( x(t) = e^{5t} - e^{3t} \) and \( y(t) = e^{3t} - 3e^{5t} \).

**Computer Assignments**

1. Use the MATLAB function `euler.m` to find the approximate solution of the IVP

\[ y' = ty^2 - \frac{y}{t}, \quad y(1) = 1 \]

over the interval \([1, 2]\) with \( h = 0.1 \)

2. Use the MATLAB function `rk24.m` to find the maximum value over the interval \([1, 2]\) of the solution of the IVP

\[ y' = \frac{1.8}{t} - y^2, \quad y(1) = -1. \]

3. The solution of the IVP

\[ y' = \frac{2}{t^2} - y^2, \quad y(1) = -0.414 \]

crosses the \( t \)-axis at a point in the interval \([1, 2]\). By experimenting with the MATLAB function `rk24.m` determine this point.
4. The solution to the IVP

\[ y' = y^2 - 2e^t + e^{2t} + e^t, \quad y(0) = 3 \]

has a vertical asymptote at some point in the interval \([0, 2]\). By experimenting with the MATLAB function \texttt{rk2.m} determine this point.

5. Use the MATLAB function \texttt{abash.m} to find the approximate solution of the IVP

\[ y' = ty^2 - \frac{y}{t}, \quad y(1) = 1 \]

over the interval \([1, 2]\) with \(h = 0.1\)

6. Solve the preceding exercise using the MATLAB function \texttt{amouson.m}

7. In the study of the nonisothermal flow of a Newtonian fluid between parallel plates, the equation

\[ y'' + \ell^2 e^y = 0, \quad t > 0, \]

was encountered. By a series of substitutions, this equation can be transformed into the first order equation

\[ \frac{dv}{du} = u \left( \frac{u}{2} + 1 \right) v^\beta + \left( u + \frac{5}{2} \right) v^\gamma. \]

Use the MATLAB function \texttt{rk2.m} to approximate \(v(3)\) if \(v(u)\) satisfies \(v(2) = 0.1\).