9.3 THEOREM. Chain Rule. If \( \mathbf{r} \) is a differentiable vector function and \( s = u(t) \) is a differentiable scalar function then the derivative of \( \mathbf{r}(s) \) with respect to \( t \) is
\[
\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds} \cdot \frac{ds}{dt}
\]

9.4 THEOREM. Rules of Differentiation. Let \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \) be differentiable vector functions and \( u(t) \) a differentiable scalar function. Then

(i) \[
\frac{d}{dt}[\mathbf{r}_1(t) + \mathbf{r}_2(t)] = \mathbf{r}_1'(t) + \mathbf{r}_2'(t);
\]

(ii) \[
\frac{d}{dt}[u(t)\mathbf{r}_1(t)] = u(t)\mathbf{r}_1'(t) + u'(t)\mathbf{r}_1(t);
\]

(iii) \[
\frac{d}{dt}[\mathbf{r}_1(t) \cdot \mathbf{r}_2(t)] = \mathbf{r}_1'(t) \cdot \mathbf{r}_2(t) + \mathbf{r}_1(t) \cdot \mathbf{r}_2'(t);
\]

(iv) \[
\frac{d}{dt}[\mathbf{r}_1(t) \times \mathbf{r}_2(t)] = \mathbf{r}_1'(t) \times \mathbf{r}_2(t) + \mathbf{r}_1(t) \times \mathbf{r}_2'(t).
\]
Integrals of Vector Functions. If \( f, g \) and \( h \) are integrable then the indefinite and definite integrals of a vector function \( \mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \) are defined, respectively, by

\[
\int \mathbf{r}(t) \, dt = \left[ \int f(t) \, dt \right] \mathbf{i} + \left[ \int g(t) \, dt \right] \mathbf{j} + \left[ \int h(t) \, dt \right] \mathbf{k}
\]

\[
\int_a^b \mathbf{r}(t) \, dt = \left[ \int_a^b f(t) \, dt \right] \mathbf{i} + \left[ \int_a^b g(t) \, dt \right] \mathbf{j} + \left[ \int_a^b h(t) \, dt \right] \mathbf{k}
\]

The indefinite integral of \( \mathbf{r}(t) \) is another vector function \( \mathbf{R}(t) + c \) such that \( \mathbf{R}(t) = \mathbf{r}(t) \).

Length of a Space Curve. If function \( \mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \) is a smooth function then it can be shown that the length of the smooth curve traced by \( \mathbf{r}(t) \) is given by

\[
s = \int_a^b \sqrt{\left[ f'(t) \right]^2 + \left[ g'(t) \right]^2 + \left[ h'(t) \right]^2} \, dt = \int_a^b \| \mathbf{r}'(t) \| \, dt.
\]

Example 8*(p.456) Integral of a Vector Function

If \( \mathbf{r}(t) = t \mathbf{i} + 3t^2 \mathbf{j} + 4t^3 \mathbf{k} \) then

\[
\int_a^b \mathbf{r}(t) \, dt = \int_a^b (t \mathbf{i} + 3t^2 \mathbf{j} + 4t^3 \mathbf{k}) \, dt
\]

\[
= \left[ \int_a^b t \, dt \right] \mathbf{i} + \left[ \int_a^b 3t^2 \, dt \right] \mathbf{j} + \left[ \int_a^b 4t^3 \, dt \right] \mathbf{k}
\]

\[
= \left[ \frac{t^2}{2} \right]_a^b + \left[ t^3 \right]_a^b + \left[ t^4 \right]_a^b
\]

\[
= \frac{3}{2} \mathbf{i} + 9 \mathbf{j} + 15 \mathbf{k}.
\]

Example 9*(pp.456-7) Length of a Space Curve

Find the length of the curve traced by \( \mathbf{r}(t) = t \mathbf{i} + t \cos t \mathbf{j} + t \sin t \mathbf{k} \) \((0 \leq t \leq \pi)\). Solution.

\[
s = \int_0^\pi \| \mathbf{r}'(t) \| \, dt = \int_0^\pi \| \mathbf{i} + (\cos t - t \sin t) \mathbf{j} + (\sin t - t \cos t) \mathbf{k} \| \, dt
\]

\[
= \int_0^\pi \sqrt{1 + (\cos t - t \sin t)^2 + (\sin t - t \cos t)^2} \, dt
\]

\[
= \int_0^\pi \sqrt{2 + t^2} \, dt
\]

\[
= \frac{\pi \sqrt{2 + \pi^2}}{2} + \sinh^{-1}(\pi \sqrt{2} / 2). \quad (Integration \ by \ parts \ then \ using \ integral \ table).
\]
9.5 Directional Derivative

The Gradient of a Function. The vector differential operator called del- or nabla- operator is given by

\[ \nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} \] or \[ \nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \]

When the del-operator is applied to a differentiable function \( z = f(x, y) \) or \( w = F(x, y, z) \), we say that the vectors

\[ \nabla f(x, y) = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j \] and \[ \nabla F(x, y, z) = \frac{\partial F}{\partial x} i + \frac{\partial F}{\partial y} j + \frac{\partial F}{\partial z} k \]

are the gradients of \( f \) and \( F \), respectively. \( \nabla f \) is usually read: grad \( f \).
9.5 Definition  Directional Derivative

The directional derivative of \( z = f(x, y) \) in the direction of a unit vector \( \mathbf{u}(\theta) = \cos \theta \mathbf{i} + \sin \theta \mathbf{j} \) is

\[
D_{\mathbf{u}} f(x, y) = \lim_{h \to 0} \frac{f(x + h\cos \theta, y + h\sin \theta) - f(x, y)}{h}
\]

provided the limit exists.

9.6 Theorem. Computing a Directional Derivative. If \( z = f(x, y) \) is a differentiable function of \( x \) and \( y \) and \( \mathbf{u} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j} \), then

\[
D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}
\]  \hspace{1cm} (5)
Maximum Value of the Directional Derivative  *Let* \( f \) *represent a function of either two or three variables. Since (5) and its three-variable analogue express the directional derivative as a dot product, we see from Definition 7.3 (MATH 201) that*

\[
D_u f = \|\nabla f\| \|u\| \cos \phi = \|\nabla f\| \cos \phi, \quad (\|u\| = 1),
\]

*where* \( \phi \) *is the angle between* \( \nabla f \) *and* \( u \). Because \( 0 \leq \phi \leq \pi \) *we have* \(-1 \leq \cos \phi \leq 1 \) *and, consequently,* \(-\|\nabla f\| \leq D_u f \leq \|\nabla f\| \). *In other words:*

*The maximum value of the directional derivative is* \( \|\nabla f\| \) *and it occurs when* \( u \) *has the same direction as* \( \nabla f \) *(when* \( \cos \phi = 1 \) *),*

*and:*

*The minimum value of the directional derivative is* \(-\|\nabla f\| \) *and it occurs when* \( u \) *and* \( \nabla f \) *have opposite directions* *(when* \( \cos \phi = -1 \) *).*

Example 4 (pp. 476-7) Directional Derivative

Example 8 (pp. 478) Direction to Cool Off Fastest
9.7 Divergence and Curl

Vector Fields  *Vector functions of two and three variables*,

\[ F(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j} \]

\[ \vec{F}(x, y, z) = P(x, y, z) \mathbf{i} + Q(x, y, z) \mathbf{j} + R(x, y, z) \mathbf{k} \]

are also called *vector fields*. 
9.7 DEFINITION  Curl  
*The Curl of a vector field* \( \mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k} \) *is the vector field*

\[
curl \mathbf{F} = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}
\]

9.8 DEFINITION  Divergence  
*The Divergence of a vector field* \( \mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k} \) *is the scalar function*

\[
\text{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.
\]
Definitions. (p. 487)

(i) If \( \text{div}(\mathbf{F}(P)) > 0 \), then \( P \) is said to be a source for \( \mathbf{F} \);
(ii) If \( \text{div}(\mathbf{F}(P)) < 0 \), then \( P \) is said to be a sink for \( \mathbf{F} \);
(iii) If \( \nabla \cdot \mathbf{F} = 0 \), the fluid is said to be incompressible;
(iv) If \( \nabla \cdot \mathbf{F} = 0 \), the vector field \( \mathbf{F} \) is said to be solenoidal.
9.8 Line Integrals

The notion of the definite integral \( \int_a^b f(x)dx \), that is, integration of a function defined over an interval, can be generalized to integration of a function defined along a curve. To this end we need some terminology about curves.

Definitions/Terminology. Suppose \( C \) is a curve parametrized by \( x = f(t), y = g(t), \ a \leq t \leq b \), and \( A \) and \( B \) are the points \( (f(a), g(a)) \) and \( (f(b), g(b)) \), respectively we say that:

(i) \( C \) is a smooth curve if \( f' \) and \( g' \) are continuous on the closed interval \([a, b]\) and not simultaneously zero on the open interval \((a, b)\);

(ii) \( C \) is piecewise smooth if it consists of a finite number of smooth curves \( C_1, C_2, \ldots, C_n \) joined end to end, that is,

\[ C = C_1 \cup C_2 \cup \ldots \cup C_n; \]

(iii) \( C \) is a closed curve if \( A = B \);

(iv) \( C \) is a simple closed curve if \( A = B \) and it does not cross itself;

(v) If \( C \) is not a closed curve, then the positive direction on \( C \) is the direction corresponding to increasing values of \( t \).

Line Integrals in the Plane \((z = G(x, y))\).
Example 1* (p. 491) Evaluation of Line Integral.

A line integral along a piecewise smooth curve $C$ is defined as the sum of the integrals over the various smooth curves whose union comprises $C$. For example, if $C$ is composed of smooth curves $C_1$ and $C_2$, then

$$\int_{C} G(x, y) ds = \int_{C_1} G(x, y) ds + \int_{C_2} G(x, y) ds$$

In many applications, line integrals appear as a sum

$$\int_{C} P(x, y) dx + \int_{C} Q(x, y) dy = \int_{C} P(x, y) dx + \int_{C} Q(x, y) dy$$

A line integral along a closed curve $C$ is very often denoted by

$$\oint_{C} P(x, y) dx + \oint_{C} Q(x, y) dy$$
Example 4* (pp. 492-3) Closed Curve Defined by an Explicit Function
Line Integrals in Space. \( w = G(x, y, z) \)
Method of Evaluation. If \( C \) is a smooth curve in 3-space defined by the
parametric equations: \( x = f(t), y = g(t), z = h(t), \ a \leq t \leq b, \) then
\[
\int_C G(x, y, z)\,dz = \int_a^b g(f(t), g(t), h(t))h'(t)\,dt.
\]
The integrals \( \int_C G(x, y, z)\,dx \) and \( \int_C G(x, y, z)\,dy \) are evaluated in a similar
fashion. The line integral WRT to arc length is
\[
\int_C G(x, y, z)\,ds = \int_a^b g(f(t), g(t), h(t))\sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} \, dt.
\]
We are often concerned with line integrals in the form of a sum:
\[
\int_C P(x, y, z)\,dx + Q(x, y, z)\,dy + R(x, y, z)\,dz
\]
Example 5* (p. 494) Line Integrals on a Curve in 3-space.
Note that since $F(x, y) \cdot dr = P(x, y)dx + Q(x, y)dy$, where $F(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ and $dr = dx\mathbf{i} + dy\mathbf{j}$, then we may express the line integral more compactly as

$$\int_c P(x, y)dx + Q(x, y)dy = \int_c F \cdot dr$$

Similarly, for a line integral on a 3-space curve,

$$\int_c P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz = \int_c F \cdot dr,$$

where $F(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ and $dr = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$.

Work. The work done by a force $F$ acting on an (massless) object moving along $C$ (with a position vector: $\mathbf{r}(t)$) is defined as

$$W = \int_c P(x, y)dx + Q(x, y)dy = \int_c F \cdot dr$$

The units of work depend on the units of $\|F\|$ and on the units of distance.

Example 6*. (p.495) Work Done by a Force
9.9 Independence of Path

The value of a line integral generally depends on the curve or path between A and B. However, there are exceptions, in other words, there are line integrals that are independent of the path between A and B.

The following concepts will be needed in the ensuing discussion:

Differential – Functions of Two Variables. The differential of a function \( \phi(x, y) \) is

\[
d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy.
\]

A differential expression \( P(x, y)dx + Q(x, y)dy \) is said to be an exact differential if there exists a function \( \varphi(x, y) \) such that

\[
d\phi = P(x, y)dx + Q(x, y)dy.
\]

For example, the expression \( x^2 y^3 dx + x^3 y^2 dy \) is an exact differential since it is the differential of \( \varphi(x, y) = \frac{1}{3}x^3 y^3 \). (Verify this!). On the other hand \( (2y^2 - 2y)dx + (2x^2 - xy)dy \) is not, or is it?

Differential – Functions of Three Variables. The differential of a function \( \phi(x, y, z) \) is

\[
d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz.
\]

A differential expression \( P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz \) is said to be an exact differential if there exists a function \( \varphi(x, y, z) \) such that

\[
d\phi = P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz.
\]

Path Independence. A line integral whose value is the same for every curve or path connecting A to B is said to be independent of the path.

Example 1*. (p.499) An Integral Independent of Path in the Plane
9.6 THEOREM. Fundamental Theorem for Line Integrals.

Suppose there exists a function \( \phi(x, y) \) such that \( d\phi = P(x, y)dx + Q(x, y)dy \): that is, \( P(x, y)dx + Q(x, y)dy \) is an exact differential. Then \( \int_C P(x, y)dx + Q(x, y)dy \) depends on only the end points \( A \) and \( B \) of the path \( C \) and

\[
\int_C P(x, y)dx + Q(x, y)dy = \phi(B) - \phi(A).
\]

\textbf{Proof.}
Notation. A line integral \(\int_c P(x, y)dx + Q(x, y)dy\), which is independent of the path between the end points A and B, is often written as \(\int_A^B Pdx + Qdy\).

Example 2*. (p.500) Using Theorem 9.8

9.9 THEOREM. Test for Path Independence in the Plane.

Let \(P\) and \(Q\) have continuous first partial derivatives in an open simply connected region. Then \(\int_c P(x, y)dx + Q(x, y)dy\) is independent of the path \(C\) if

\[
\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}
\]

for all \((x, y)\) in the region.

Example 3*. (p.501) A Path-Dependent integral
Example 4*. (p.501) An integral independent of the Path

Assignment. Extract the definitions of the following terms: gradient field, potential function and conservative force field.
9.10 THEOREM. Test for Path Independence in Space.

Let $P$, $Q$ and $R$ have continuous first partial derivatives in an open simply connected region of space. Then

\[ \int_C (P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz) \]

is independent of the path $C$ if and only if

\[ \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial z}, \quad \frac{\partial P}{\partial x} = \frac{\partial R}{\partial z}, \quad \frac{\partial Q}{\partial x} = \frac{\partial R}{\partial y} \]

for all $(x, y, z)$ in the region.

Example 6*. (p.503) An integral independent of the Path
9.12 Green's Theorem

Introduction. *One of the most important theorems in vector integral calculus relates a line integral around a piecewise smooth simple closed curve to a double integral over the region R bounded by the curve.*

Line Integrals Along Simple Closed Curves. *We say that the positive direction around a simple closed curve C is that direction a point on the curve must move, or the direction a person must walk on C, in order to keep the region R bounded by C to the left. See the Figures on page 519. Note also the notations for line integrals in the positive and negative directions, respectively.*

9.13 THEOREM. Green's Theorem in the Plane.

Suppose that C is a piecewise smooth simple closed curve bounding a region R. If P, Q, \( \frac{\partial P}{\partial y} \), and \( \frac{\partial Q}{\partial x} \) are continuous on R, then

\[
\oint_C P\,dx + Q\,dy = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \,dA.
\]

*Proof: Read a partial proof on page 519.*
Example 1*. (p.520) Using Green's Theorem

Example 3*. (p.521) Work done by a Force
Example 4*. (p.521) Green's Theorem Not applicable
9.13 Surface Integrals

Introduction. In the xy-plane, the length of an arc of the graph of \( y = f(x) \) from \( x = a \) to \( x = b \) is given by the definite integral

\[
s = \int_{a}^{b} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx
\]

The corresponding problem in 3-space is to find the area \( A(s) \) of that portion of the surface \( S \) given by a function \( z = f(x, y) \) having continuous first partial derivatives on a closed region \( R \) in the xy-plane. Such a surface is said to be smooth.

9.11 Definition. Surface Area

Let \( f \) be a function for which the first partial derivatives \( f_x \) and \( f_y \) are continuous on a closed region \( R \). Then the area of the surface over \( R \) is given by

\[
A(S) = \iint_{R} \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2} \, dA
\]

Example 1*. (p.526) Surface Area.
Differential of Surface Area  

The function 

\[ dS = \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2} \, dA \]

is called the differential of the surface area.

Evaluating Surface Integrals  

Let \( w = G(x, y, z) \) and \( z = f(x, y) \). If \( G, f, f_x, \) and \( f_y \) are continuous throughout a region containing \( S \) then

\[
\int_S G(x, y, z) \, dS = \int_R G(x, y, f(x, y))\sqrt{1 + [f_x]^2 + [f_y]^2} \, dA.
\]

Note that when \( G = 1 \) the above formula reduces to that for surface area.

Projection of \( S \) Into Other Planes  

If \( y = g(x, z) \) is the equation of the surface \( S \) that projects onto a region \( R \) of the \( xz \)-plane, then

\[
\int_S G(x, y, z) \, dS = \int_R G(x, g(x, z), z)\sqrt{1 + [g_x]^2 + [g_z]^2} \, dA.
\]

Similarly, if \( x = h(y, z) \) is the equation of the surface \( S \) that projects onto a region \( R \) of the \( yz \)-plane, then

\[
\int_S G(x, y, z) \, dS = \int_R G(h(y, z), y, z)\sqrt{1 + [h_y]^2 + [h_z]^2} \, dA.
\]

Mass of a Surface  

Suppose \( \rho(x, y, z) \) represents the density of a surface at any point, or mass per unit surface area; then mass \( m \) of the surface is given by

\[
m = \int_S \rho(x, y, z) \, dS
\]

Example 2*. (p.527) Mass of a Surface.
9.14 Stokes' Theorem

Introduction. Green's Theorem in the previous section (Section 9.12) has two vector forms. In this section and the next (Section 9.16) we shall generalize these forms to three dimensions.

Vector Form of Green's Theorem. If \( \mathbf{F}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j} \) is a 2-space vector field, then

\[
\text{Curl} \, \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}
\]

From (12) and (13) on page 495, Green's Theorem can be written in vector notation as

\[
\oint \mathbf{F} \cdot d\mathbf{r} = \iint (\text{Curl} \, \mathbf{F} \cdot \mathbf{k}) \, dA,
\]

that is, the line integral of the tangential component of \( \mathbf{F} \) is the double integral of the normal component of \( \text{Curl} \, \mathbf{F} \).

9.14 THEOREM. Stokes' Theorem (Green's Theorem in 3-Space). Let \( S \) be a piecewise smooth orientable surface bounded by a piecewise smooth simple closed curve \( C \). Let

\[
\mathbf{F}(x, y, z) = P(x, y, z) \mathbf{i} + Q(x, y, z) \mathbf{j} + R(x, y, z) \mathbf{k}
\]

be a vector field for which \( P, Q \) and \( R \) are continuous and have continuous first partial derivatives in a region of 3-space containing \( S \). If \( C \) is traversed in the positive direction, then

\[
\oint \mathbf{F} \cdot d\mathbf{r} = \iint_{S} (\text{Curl} \, \mathbf{F}) \cdot \mathbf{n} \, dS,
\]

where \( \mathbf{n} \) is a unit normal to \( S \) in the direction of the orientation of \( S \).
Example 1*. (pp.534-6) Verifying Stokes' Theorem
Example 2*. (pp.536) Using Stokes' Theorem
9.16 Divergence Theorem

Introduction. *In this section we present a second form of Green's Theorem.*

Another Vector Form of Green's Theorem. Let \( \mathbf{F}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j} \) be a 2-space vector field, and let \( \mathbf{T} = (dx/ds) \mathbf{i} + (dy/ds) \mathbf{j} \) be a unit tangent to a simple closed plane curve \( C \). In Section 9.4 we saw that \( \oint_C (\mathbf{F} \cdot \mathbf{T}) \, ds \) can be evaluated by a double integral involving \( \text{curl} \, \mathbf{F} \). Similarly, if \( \mathbf{n} = (dy/ds) \mathbf{i} - (dx/ds) \mathbf{j} \) is a unit normal to \( C \), then \( \oint_C (\mathbf{F} \cdot \mathbf{n}) \, ds \) can be expressed in terms of a double integral of \( \text{div} \, \mathbf{F} \). From Green's Theorem,

\[
\delta A \frac{d\mathbf{F}}{dy} - Q \, d\delta A = \oint_C \left[ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] \, ds,
\]

that is,

\[
\oint_C (\mathbf{F} \cdot \mathbf{n}) \, ds = \iint_D \text{div} \, \mathbf{F} \, d\mathbf{A}.
\]

This result is a special case of the divergence or Gauss' theorem below.

9.15 THEOREM. Divergence Theorem (Gauss' Theorem in 3-Space). Let \( D \) be a closed and bounded region in 3-space with a piecewise smooth boundary \( S \) that is oriented outward. Let \( \mathbf{F}(x, y, z) = P(x, y, z) \mathbf{i} + Q(x, y, z) \mathbf{j} + R(x, y, z) \mathbf{k} \) be a vector field for which \( P, Q \) and \( R \) are continuous and have continuous first partial derivatives in a region of 3-space containing \( D \). Then

\[
\iint_D \text{div} \, \mathbf{F} \, dV = \iint_S (\mathbf{F} \cdot \mathbf{n}) \, dS
\]

where \( \mathbf{n} \) is a unit normal to \( S \) in the direction of the orientation of \( S \).

*Proof. Read a partial proof on pages 550-2.*

Example 1*. (pp.552-3) Verifying Divergence Theorem

Example 2*. (p.553) Using Divergence Theorem