Pointwise Convergence of Wavelet Expansions Associated with Dilation Matrix

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The wavelet expansion associated with dilation matrix of a function is studied. This expansion is shown to converge uniformly on compact subsets.

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1. INTRODUCTION

There has been rapid growth in the theory and applications of wavelets [2, 6] in recent years. An interesting introduction of the wavelets is given by Daubechies [3]. Kelly et al [4, 5] and Walter [7] have studied the convergence of wavelet expansions. A comprehensive discussion on two-dimensional
wavelet expansions incorporating rotation can be found in Antoine et al. [1]. A detailed account of multidimensional multiresolution analysis is presented by Wojtaszczyk [8]. In this paper we study the convergence of multiwavelet expansion associated with the multiresolution analysis with dilation matrix. Our theorem is a generalization of Walter's results. However analogous results to Theorems 3.4 and 3.7 [4] require further investigation and will be presented separately. The main theorem of Kelly et al. is analogous to the famous Carleson result of Fourier series; Fourier series was shown to converge almost everywhere, but not for all Lebesgue points. In their theorem, pointwise convergence results state that, with few conditions on the wavelets, a wavelet expansion for a function \( f \in L_p(\mathbb{R}^d) \), \( d \geq 1 \), converges on the Lebesgue set of \( f \). This result is a general form of Walter's result, (Theorem 1 and Corollary 1 [7]).

2. MULTIDIMENSIONAL MULTiresOLUTION ANALYSIS

Let \( A \) be any real expansive \( n \times n \) matrix (equivalently, all eigenvalues of \( A \) are required to have absolute value \( > 1 \)). A wavelet set associated with the matrix \( A \), called dilation matrix, is a finite set of functions \( \psi^r(x) \in L_2(\mathbb{R}^d), \quad r = 1, 2, 3, \ldots, s \) such that the system

\[
\{ |\det A|^{r/2} \psi^r(A^j x - \gamma) \}
\]

with \( r = 1, 2, \ldots, s \), \( j \in \mathbb{Z} \) and \( \gamma \in \mathbb{Z}^d \) (\( \mathbb{Z} \) denotes a set of positive integers) is an orthonormal basis in \( L_2(\mathbb{R}^d) \). It is a generalization of the notion of wavelet. By analogy with the one-dimensional case, we may use the notation: for a function \( F(\varphi, \psi, \text{etc.}) \) on \( \mathbb{R}^d \), by \( F_{j, \gamma} \), we mean

\[
F_{j, \gamma}(x) = |\det A|^{j/2} F(A^j x - \gamma), \quad \text{where} \quad j \in \mathbb{Z}, \quad \text{and} \quad \gamma \in \mathbb{Z}^d.
\]

We shall omit \( r \) as there is no ambiguity.

A multiresolution associated with dilation matrix \( A \) is a sequence of closed subspaces \( \{V_j\}_{j \in \mathbb{Z}} \) of \( L_2(\mathbb{R}^d) \) satisfying

(i) \( \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \),

(ii) \( \bigcup_{j \in \mathbb{Z}} V_j \) is dense in \( L_2(\mathbb{R}^d) \),

(iii) \( \bigcap_{j \in \mathbb{Z}} V_j = \{0\} \).

(iv) \( f \in V_j \) if and only if \( f(Ax) \in V_{j+1} \), that is, \( V_j = U_j V_0 \),

(v) \( f \in V_0 \) if and only if \( f(x - \gamma) \in V_0 \) for all \( \gamma \in \mathbb{Z}^d \), and
there exists a function $\varphi \in V_0$ called a scaling function such that the system \( \{ \varphi(t - \gamma) \}_{\gamma \in \mathbb{Z}^d} \) is an orthonormal basis in \( V_0 \).

The following results are relevant to our discussion:

**Theorem A** [8, p. 116]. For every multiresolution on \( \mathbb{R}^d \) associated with a dilation matrix \( A \), there exists an associated wavelet set (consisting of \( q - 1 \) functions, where \( q = |\det A| \)).

A function \( F \) on \( \mathbb{R}^d \) is called \( r \)-regular if \( F \) is of class \( C^r \), \( r = -1, 0, 1, \ldots \) and

$$\frac{\partial^\alpha F(x)}{\partial x^\alpha} \leq \frac{C_k}{(1 + |x|)^k}$$

for each \( k = 0, 1, 2, \ldots \) and each multi-index \( \alpha \) with \( |\alpha| \leq \max(r, 0) \) and some constant \( C_k \). As usual, \( C^{-1} \) means a measurable function and class \( C^0 \) means a function.

A multiresolution analysis on \( \mathbb{R}^d \) is called \( r \)-regular if it has an \( r \)-regular scaling function.

**Theorem B** [8, p. 118]. For every \( r \)-regular multiresolution analysis on \( \mathbb{R}^d \) associated with a dilation matrix \( A \), \( |\det A| = q \), such that \( 2q - 1 > d \), there exists an associated wavelet set consisting of \( q - 1 \) \( r \)-regular functions.

**Corollary A** [8, p. 120]. Assume that we have a multiresolution on \( \mathbb{R}^d \) associated with a dilation matrix \( A \), \( |\det A| = q \). Assume further that this multiresolution analysis has an \( r \)-regular scaling function \( \varphi(x) \) such that its Fourier transform \( \hat{\varphi}(s) \) is real. Then there exists a wavelet set associated with this multiresolution analysis consisting of \( q - 1 \) \( r \)-regular functions.

**Theorem C** [8, p. 136]. Suppose \( A \) is a dilation matrix such that for some set of digits \( S = \{ k_1, k_2, \ldots, k_q \} \) a subset of \( \mathbb{R}^d \), \( Q \) defined by

$$Q = \{ x \in \mathbb{R}^d : x = \sum_{j=1}^{\infty} A^{-j}s_j, \text{ where } s_j \in S \}$$

has measure 1, that is, the characteristic function \( \chi_Q \) of \( Q \) is a scaling function of a multiresolution analysis. Then for each natural number \( r = 1, 2, 3, \ldots \) there exists an \( r \)-regular wavelet set (consisting of \( |\det A| - 1 \) functions) associated with the dilation matrix \( A \).

**Examples** [8, pp. 127–129].

(i) Let us take \( d = 2 \) and the simplest dilation \( A = 2I_d \). Taking the set of digits \( S \) as \( \{(0,0),(0,1),(1,0),(1,1)\} \), we get \( Q = [0, 1]^2 \). This clearly gives scaling function of a multiresolution analysis. Choosing the set \( S \) as

$$\{(0,0),(1,1),(0,1),(1,2)\},$$

we obtain as the set \( Q \) the parallelogram with vertices from the set \( S \). This also gives a scaling function of a multiresolution analysis. We take \( S = \{(0,0),(1,0),(0,1),(-1,-1)\} \), then \( Q \) will be like the Sierpinski triangle (see Figure 5.3 in [8, p. 129]).
(ii) Let us take \( d = 2 \) and the dilation given by the matrix \[
\begin{bmatrix}
1 & -1 \\
1 & 1
\end{bmatrix}
\]. Geometrically speaking, this dilation is a rotation by \( 45^\circ \) and expansion by the factor \( \sqrt{2} \). Since \( \det A = 2 \), we obtain one wavelet generating an orthonormal basis in \( L_2(\mathbb{R}^2) \) provided there is a scaling function. Taking \( S = \{(0,0),(1,0)\} \), we get \( Q \) as the fractal set known as the 'twin dragon' [8, pp. 129-130].

3. WAVELET EXPANSION ASSOCIATED WITH DILATION MATRIX

Associated with the \( V_j \) spaces in the definition of multiresolution analysis, there is the orthogonal complement of \( V_j \) in \( V_{j+1} \), denoted by \( W_j \) such that \( V_{j+1} = V_j \oplus W_j \). Thus \( L_2(\mathbb{R}^d) = \sum \oplus W_j \). We define \( P_j \) and \( \bar{Q}_j = P_{j+1} - P_j \), respectively, to be the orthogonal projections onto the spaces \( V_j \) and \( W_j \), with kernels \( P_j(x,y) \) and \( \bar{Q}_j(x,y) \). By Theorem A there exists an associated wavelet set consisting of \( q-1 \) functions, where \( q = |\det A| \). Corollary A guarantees the existence of \( r \)-regular wavelet sets. The sequence of projections \( \{P_j f(x)\} \) is called the multiresolution expansion of \( f \). The scaling expansion of \( f \) is defined as

\[
f \sim \sum_{\gamma} b_{j,\gamma} \Delta_j \varphi (A^j x - \gamma) + \sum_{k=0}^{\infty} \alpha_{k,\gamma} A^k \psi (A^k x - \gamma) \quad (3.1)
\]

where

\[
a_{j,\gamma} = \int_{\mathbb{R}^d} f(x) F_{j,\gamma}(x) dx \quad (3.2)
\]

\[
b_{j,\gamma} = \int_{\mathbb{R}^d} f(x) |\det A|^j \varphi (A^j x - \gamma) dx, \quad (3.3)
\]

and \( f \in L_2(\mathbb{R}^d) \).

The wavelet expansion associated with dilation matrix \( A \) or multivariable wavelet expansion of \( f \) is

\[
f \sim \sum_{j,\gamma} a_{j,\gamma} F_{j,\gamma}(x) dx \quad (3.4)
\]

where \( a_{j,\gamma} \) is given in (3.2).

Considering convergence in the sense of \( L_2(\mathbb{R}^d) \), we may write

\[
f(t) = \sum_{j} \sum_{\gamma} a_{j,\gamma} F_{j,\gamma}(t) \quad (3.5)
\]

and

\[
f(t) = \sum_{\gamma} b_{j,\gamma} |\det A|^{j/2} \varphi (A^j t - \gamma) + \sum_{k=0}^{\infty} \alpha_{k,\gamma} F_{k,\gamma}(t) = f_j(t) + r_m(t). \quad (3.6)
\]
The function \( f_m \in V_m \) is, in fact, the projection \( f \) onto \( V_m \). It can be written as

\[
    f_m(x) = \int_{\mathbb{R}^d} q_m(x, t)f(t)dt
\]

where

\[
    q_m(x, t) = |\det A|^{m/2} q(A^m x, A^m t)
\]

and

\[
    q(x, t) = \sum_{\gamma} \varphi(x - \gamma)\varphi(t - \gamma), \quad \gamma \in \mathbb{Z}^d,
\]

\( q_m(x, t) \) will be called the reproducing kernel of \( V_m \).

We will use mainly the following results for wavelet expansion given in (3.4).

For a scaling function \( \varphi \) associated with a dilation matrix considered in Section 1, the following results hold [8, p. 138].

\[
    \int_{\mathbb{R}^d} \varphi(x)dx = 1
\]

\[
    \sum_{\gamma \in \mathbb{Z}^d} \varphi(x - \gamma) = 1
\]

\[
    |\varphi(t)| \leq \frac{C_k}{(1 + |t|)^k}, \quad k = 1, 2, 3, \ldots
\]

A sequence \( \delta_m(x, y) \) of functions in \( L_1(\mathbb{R}^d) \) is called a quasi-positive delta sequence if the following conditions are satisfied:

there exists a constant \( C \) such that

\[
    \int_{\mathbb{R}^d} \delta_m(x, y)dy \leq C, \quad \text{for all } y \in \mathbb{R}^d, \quad m \in \mathbb{N}
\]

there exists a vector \( c = (c_1, c_2, \ldots, c_d) > 0 \) such that

\[
    \int_{y \cdot c + \mathbb{Z}^d} \delta_m(x, y)dx \rightarrow 1
\]

uniformly on compact subset of \( \mathbb{R}^d \) as \( m \rightarrow \infty \);

for each \( r > 0, \)

\[
    \sup_{|x-y| \leq r} |\delta_m(x, y)| \rightarrow 0 \quad \text{as } m \rightarrow \infty.
\]

**LEMMA 3.1.** The reproducing kernel \( q_m(x, y) \) of \( V_m \), the multiresolution analysis associated with a dilation matrix \( A \), is a quasi-delta sequence.

For the sake of convenience we write the proof for the two-dimensional case.

**Proof of Lemma 3.1.** We have

\[
    \int_{\mathbb{R}^2} |q_m(x, y)|dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\det A|^{m/2} |q(A^m x, A^m y)|dx
\]

\[
    = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |q(x, A^m y)|dx
\]

\[
    \leq C_k \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 + |x - A^m y|)^{-k}dx = C
\]

by (3.12).
Thus (3.13i) holds.

We can write

\[
\int_{y-c}^{y+c} \int_{y-c}^{y+c} q_m(x, y) dx = \int_{A_m^c}^{A_m^c(y+c)} \int_{A_m^c}^{A_m^c(y+c)} q(x, A_m^c y) dx
\]

\[
\leq \int_{t-A_m^c(y+c)}^{t-A_m^c(y+c)} q(x, t) dx
\]

\[
= 1 - \int_{t-A_m^c(y+c)}^{t+A_m^c(y+c)} q(x, t) dx - \int_{t+A_m^c(y+c)}^{t-A_m^c(y+c)} q(x, t) dx
\]

\[
= 1 - I_1 - I_2
\]

\[
I_1 \leq c \int_{t+A_m^c(y+c)}^{t+A_m^c(y+c)} \frac{1}{1 + (t-x)^k} dx
\]

\[
= c \int_{A_m^c(y+c)}^{A_m^c(y+c)} \frac{1}{1 + x^k} dx \to 0, \quad k > 1,
\]

as \( m \to \infty \). Similarly, \( I_2 \to 0 \) as \( m \to \infty \). Hence (3.13ii) holds. (3.13iii) can also be verified by using (3.12).

4. CONVERGENCE THEOREM

In 1966, Carleson proved the famous Lusin conjecture that the Fourier series of an arbitrary \( L_2(R) \) function \( f \) converges pointwise almost everywhere to \( f \). This result was extended by Hunt to \( L_p \) functions when \( 1 < p < 2 \). In 1971, C. Fefferman proved that spherically summed two-dimensional Fourier series of \( L_p(R^2) \) functions do not converge in \( L_p(R^2) \) for certain real \( p \). Kelly, Kon, Raphael [4, 5] have studied the convergence of wavelet expansions indicating the inter-connection between classical results in this area and their results including those concerning such expansions by Meyer and Walter.

We prove here a theorem on the pointwise convergence of two-dimensional wavelet expansions associated with a dilation matrix. The proof is also valid for higher dimensions. More precisely, we prove that a wavelet expansion associated with a dilation matrix of a continuous function \( f \) belonging to \( L_1(R^2) \cap L_2(R^2) \) converges uniformly on a compact subset. It may be observed that this result extends Lemma 1, Corollary 1 and Theorem 1 in Walter [7].

Relaxation of continuity and weakening of regularity condition on the wavelet require further investigation on the lines of Kelly, Kon and Raphael [5].

THEOREM 4.1. Let \( q_m(x, y) \) be a reproducing kernel of a multiresolution analysis associated with a dilation matrix \( A \); and let \( f \in L_1(R^2) \) be continuous on an open set \( U \) in \( R^2 \), then

\[
f_m(y) = \int_{[y-n,y+n] \times R} q_m(x, y) f(x) dx \to f(y) \quad (4.1)
\]
as $m \to \infty$ uniformly on compact subsets of $U$.

**COROLLARY 4.1.** Let $f \in L_1(R^2) \cap L_2(R^2)$ be continuous on a subset $U$ and let $f_m$ be the projection of $f$ into $V_m$, then

$$f_m \to f \text{ as } m \to \infty,$$

uniformly on compact subsets of $U$.

**Proof of Theorem 4.1.** Let $\eta > 0$, then

$$f_m(y) = \int_{[y_n,y_n+n]} q_m(x,y) f(x) dx + \int_{[y_{n+\infty},y_n+\infty]} q_m(x,y) f(x) dx + \int_{[-\infty,y_n-n]} q_m(x,y) f(x) dx$$

$$= f(y) \int_{[y_n,y_n+n]} q_m(x,y) dx + \int_{[y_{n+\infty},y_n+\infty]} q_m(x,y)(f(x) - f(y)) dx + \left\{ \int_{[y_{n+\infty},y_n+\infty]} + \int_{[-\infty,y_n-n]} \right\} = I_1 + I_2 + I_3. \quad (4.2)$$

Now let $K$ be a compact subset of $U$, and let $V$ be a closed subset contained in $U$ containing $K$. For $y \in V$, choose $\eta$ such that $0 < \eta < c$. Further, we restrict $\eta$ such that $|f(x) - f(y)| < \epsilon$ for $y \in K$ and $|x - y| < \eta$. From this it follows that

$$|I_2| \leq \epsilon \int_{[y_n,y_n+n]} |q_m(x,y)| dx \quad (4.3)$$

and

$$|I_3| \leq \sup_{n \leq |x-y|} |q_m(x,y)||f|_{L_1(R^2)} \quad \text{whenever } m \geq M_1 \quad (4.4)$$

where $M_1$ is so large that

$$\sup_{n \leq |x-y|} |q_m(x,y)| < \epsilon \quad \text{for } m \geq M_1.$$

We choose $M_2 \geq M_1$ so large that

$$\left| 1 - \int_{[y_n,y_n+n]} q_m(x,y) \right| < \epsilon, \quad \text{whenever } m \geq M_2. \quad (4.5)$$

This holds because $q_m(x,y)$ is a quasi-positive delta sequence and so it follows by (3.13ii).

By (4.3), (4.4) and (4.5), we get

$$|f(y) - f_m(y)| \leq |f(y) - I_1| + |I_2| + |I_3|$$

$$\leq |f(y)| \left( 1 - \int_{y-n}^{y+\eta} q_m(x,y) dx \right)$$

$$+ \epsilon \int_{y-n}^{\infty} q_m(x,y) dx + \epsilon \|f\|_1$$

$$\leq \sup_{y \in [a,b]} |f(y)| \epsilon + \epsilon C + \epsilon \|f\|_1.$$
for $m \geq M_2$, which gives us the desired uniform convergence on $[\alpha, \beta]$ and hence on $K$. This proves the theorem.

Corollary 4.1 follows from Theorem 4.1 and Lemma 3.1

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