ON SOME RECENT DEVELOPMENTS CONCERNING MOREAU'S SWEEPING PROCESS

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Abstract

The main objective of this paper is to present an overview of Moreau’s sweeping process \( \Phi(t, \cdot, \cdot, \cdot) \) along with some of the results concerning new variants of the process. Several open problems are mentioned.

Keywords: Sweeping process, evolution variational inequality, evolution quasi-variational inequality, state-dependent sweeping process, variational sweeping process, variants of sweeping process, play and stop operation, sweeping process without continuity.

1. Introduction

A sweeping process comprises two important ingredients: one part that sweeps and the other part is swept. For example, imagine the Euclidean plane and consider a large ring with a small ball inside it; the ring starts to move at time \( t = 0 \). Depending on the motion of the ring, the ball will first stay where it is (in case it is not hit by the ring); otherwise it is swept towards the interior of the ring. In this latter case, the velocity of the ball has to point inward to the ring in order not to leave.

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We consider the case where the small ball has diameter zero, that is, it
degenerates to a point. We replace the ring and its interior by an arbitrary
convex closed set in. In mathematical terms, the problem then becomes
\[ u(t) \in N_{C(t)}(u(t)) \quad \text{as in } [0, T], \]
\[ u(0) = u_0 \in C(0). \quad (1.1) \]

Here, for any closed convex set \( C \) subset of a Hilbert space \( H \) and \( x \in C \), the set
\[ N_{C}(x) = \{ y \in H \mid \langle y, x - z \rangle \leq 0 \text{ for all } z \in C \} \quad (1.2) \]
denotes the outward normal cone to \( C \) at \( x \). \( u(t) \) denotes the position of the ball
at time \( t \) and \( C(t) \) is the ring at time \( t \). The expression \( N_{C(t)}(u(t)) \) denotes the
outward normal cone to the set \( C(t) \) at position \( u(t) \) as defined in Equation (1.2).
Thus, Equation (1.1) simply means that the velocity \( u(t) \) of the ball has to point
inward to the ring at almost every point \( t \in [0, T] \). The restriction is due to
the fact that usually we will not have a smooth function \( t \mapsto u(t) \) satisfying
(1.1), but functions satisfying (1.1) that are differentiable everywhere besides
on some subset of \( [0, T] \) of measure zero. The initial condition \( u(0) \in C(0) \) states
that the ball is initially contained in the ring. Equation (1.1) is the simplest
instance of the sweeping process, introduced by Moreau [28] in the seventies.

In general, the time-dependent moving set at \( t \) \( C(t) \) is given, and we want
to prove the existence of a solution (preferably unique) \( t \mapsto u(t) \) that will take
values in some Hilbert space (Here \( H = \mathbb{R}^2 \)). It is allowed that \( C(t) \) changes
its shape while moving, whereas in the introductory example the ring simply
moved by translation and maintained its original shape. The sweeping process
plays an important role in elastoplasticity and dynamics for unilateral problems
(see, for example, [4, 21, 35, 27, 30]).

In Section 2, we present a review of some important results for the sweeping
process (1.1) concerning existence and uniqueness of solutions. Section 3 is
dedicated to a generalization of the sweeping process where the moving set
depends on the current state \( u(t) \); that is, \( C = \{ t, u(t) \} \) instead of \( C = C(t) \).
This has been studied by Kunze and Monteiro Marques [18]. We present in
Section 4 a degenerate sweeping process studied by Kunze and Monteiro Marques
[16, 17]. In Section 5, we discuss some unpublished results of Mancho and
Südlik [23] and Südlik, Marchand and Brokate [36]. Section 6 deals with the
existence of solutions to the nonconvex sweeping case, that is, the case
when \( C(t) \) is not a convex set in \( [1, T] \). These existence results have been ob-
tained by Benabdellah [2] and Colombini and Goncharov [8]. Section 7 provides
the relationship between the play operator, the stop operator and the sweeping
process. In Section 8, we remark on several open problems.

2. Moreau’s Sweeping Process

Let \( H \) be a separable Hilbert space, let \( x \in H \) and \( C \subset H \) be closed, convex
and nonempty. Then there exists a unique \( y \in C \) that minimizes the distance of
\( x \) to \( C \), \( y \) is called the projection of \( x \) onto \( C \) and is written as \( y = \text{Proj}(x, C) \). If \( y = \text{Proj}(x, C) \) if and only if (we denote the norm in \( H \) by \( \| \cdot \| \))

\[
\| x - y \| = \text{dist}(x, C) \text{ where } \text{dist}(x, C) = \inf_{z \in C} \| x - z \| .
\]

Equivalently, \( y = \text{Proj}(x, C) \) if and only if

\[
x \in C, \quad (x - y, y - z) \leq 0 \quad \text{for all } z \in C.
\]  

(2.1)

Let us denote by

\[
d_H(C_1, C_2) = \max \left\{ \sup_{x \in C_1} \| x - y \|, \sup_{y \in C_2} \| x - y \| \right\}
\]  

(2.2)

the Hausdorff distance between the subsets \( C_1 \) and \( C_2 \) of the Hilbert space \( H \).

The variation of a function \( u : [0, T] \to H \) is defined as

\[
\text{Var}(u) = \text{Var}(u, [0, T]) = \sup \left\{ \sum_{i=0}^{N-1} \| u(t_{i+1}) - u(t_i) \| \right\},
\]

(2.3)

where \( \{ t_i \} \) is a partition of \([0, T]\).

\text{Var}(u) < \infty \quad \text{if and only if} \quad u \text{ is absolutely continuous}.

(2.4)

It can be easily checked that every Lipschitz continuous function \( u \) is of bounded variation, and that \( \text{Var}(u) \leq K T \) if (2.4) holds. \( u \) is called absolutely continuous if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that

\[
\sum_{i=1}^{N} \| u(t_{i+1}) - u(t_i) \| < \varepsilon \quad \Rightarrow \quad \sum_{i=1}^{N} \| u(t_{i+1}) - u(t_i) \| < \varepsilon
\]

holds for every finite collection \( \{ t_i \} \subset \mathbb{R} \). \( \mathbb{N} \), of pairwise disjoint non-empty subintervals \( I_k = [t_k, t_{k+1}] \) of \([0, T]\). If \( u \) is absolutely continuous, it is differentiable almost everywhere and satisfies

\[
u(t) = u(0) + \int_{0}^{t} u'(s) \, ds, \quad t \in [0, T], \quad \text{Var}(u) = \int_{0}^{T} |u'(s)| \, ds,
\]

where the integral has the meaning of the Bochner-Lebesgue integral. The choice \( \delta = \varepsilon/K \) shows that \( u \) is absolutely continuous if it satisfies (2.4).
In order to prove the existence of solutions to the sweeping process and its variants, we often construct a sequence of approximating solutions whose variations are uniformly bounded. Then the compactness result of Theorem 2.2 allows us to select a subsequence which converges towards some limit function, and then the task is to show that the limit function is indeed the desired solution.

**Definition 2.1.** An absolutely continuous function \( n : [0, T] \to H \) is a solution of the sweeping process (1.1) if

1. \( n(0) = u_0 \),
2. \( n(t) \in C(t) \) for all \( t \in [0, T] \),
3. \(-u(t) \in \mathcal{N}_C(t)(u(t))\) a.e. in \([0, T] \).

**Theorem 2.1 (Existence Theorem for Moreau's Process).** Let \( R := C(t) \) be Lipschitz continuous, that is,

\[
\text{dist} \left( C(t), C(s) \right) \leq K|t - s|, \quad t, s \in [0, T].
\]

and \( C(0) = H \) be nonempty, closed, and convex for every \( t \in [0, T] \). Let \( u_0 \in C(0) \). Then there exists a solution \( n : [0, T] \to H \) of (1.1) satisfying (2.4). In particular, \( |n(t)| \leq K \) for almost every \( t \in [0, T] \).

The following results are required for the proof of this theorem and for the subsequent discussions.

**Theorem 2.2.** Let \( H \) be a Hilbert space and \( \{u_n\} \) a sequence of functions \( u_n : [0, T] \to H \) which are bounded uniformly in norm and variation, i.e.,

\[
|u_n(t)| \leq M_1, \quad n \in \mathbb{N}, \quad t \in [0, T], \quad \text{and} \quad \text{Var}(u_n) \leq M_2, \quad n \in \mathbb{N}
\]

for some constants \( M_1, M_2 > 0 \) independently of \( n \in \mathbb{N} \) and \( t \in [0, T] \). Then there exists a subsequence \( \{u_{n_k}\}_{k=1}^{\infty} \) and a function \( u : [0, T] \to H \) such that \( \text{Var}(u) \leq M_2 \) and \( u_{n_k}(t) \to u(t) \) weakly in \( H \) for all \( t \in [0, T] \), i.e.,

\[
(u_{n_k}(t), z) \to (u(t), z) \quad \text{for all} \ z \in H \text{ as} \ k \to \infty.
\]

The following lemma summarizes some facts related to weak convergence.

**Lemma 2.1.** Let \( u_n \to u \) weakly in \( H \).

(a) \( \|u_n\| \leq \limsup_{n \to \infty} \|u_n\| \) holds.

(b) If \( u_n \in C + \overline{B}(0, R) \) for some closed convex \( C \subset H \) and some sequence \( \epsilon_n \to 0 \), then \( u \in C \).
Lemma 2.2 Let \( a : [0, T] \rightarrow H \) be an absolutely continuous function. Then
\[
\int_0^t |a'(t), u(t)| \, dt = \frac{1}{2} |u(T)|^2 - \frac{1}{2} |u(0)|^2.
\]
For more details of the proof of Theorem 2.1 and the intermediate results above, we refer to [21, 25] or [28].

Theorem 2.3 (Uniqueness of solution of Moreau's sweeping process) The solution of (1.1) is unique in the class of absolutely continuous functions.

Corollary 2.1 (Dependence on data) Under the assumptions of Theorems 2.1 and 2.3, if \( u \) and \( v \) are two solutions with \( u(0) = u_0 \) and \( v(0) = v_0 \), then
\[
|u(t) - v(t)| \leq |u_0 - v_0|, \quad t \in [0, T].
\]

Theorem 2.4 (Dependence on the moving set) Let \( t \mapsto C(t) \) and \( t \mapsto D(t) \) be two moving sets which satisfy (2.5) with Lipschitz constants \( L_C \) and \( L_D \), respectively. Assume that \( C(t) \) and \( D(t) \) are nonempty, closed, and convex for every \( t \in [0, T] \). Then, if \( s(t) \) denotes the solution to the sweeping process with \( t \mapsto C(t) \) and initial value \( u(0) = u_0 \), and if \( v(t) \) denotes the solution to the sweeping process with \( t \mapsto D(t) \) and initial value \( v(0) = v_0 \), the estimate
\[
|u(t) - v(t)|^2 \leq |u_0 - v_0|^2 + 2(L_C - L_D) \int_0^t \Delta(s)x^t. \quad t \in [0, T], \quad (2.7)
\]
holds, where
\[
\Delta(t) = \mathcal{A}_H \left( C(t), D(t) \right), \quad t \in [0, T].
\]

Proof. (See [21].) For fixed \( t \in [0, T] \), we have \( u(t) \in C(t) \subseteq D(t) + \mathcal{B}_H(0) \). Hence, there exist vectors \( d(t) \in D(t) \) and \( r(t) \in H \) such that
\[
u(t) = d(t) + r(t) \quad \text{and} \quad r(t) \in \Delta(t).
\]
It can be shown that it is possible to choose the maps \( t \mapsto d(t) \) and \( t \mapsto r(t) \) as being measurable. Similarly, we find \( v(t) = e(t) + s(t) \) with \( e(t) \in C(t) \) and \( s(t) \in \Delta(t) \). We verify using Lemma 2.2 that
\[
\frac{1}{2} \left( u(t) - v(t) \right)^2 = \left( a(t), u(t) - v(t) \right)\frac{1}{2} \left( u(t) - v(t) \right)^2 \leq (a(t), u(t) - v(t)) - \left( a(t), e(t) - v(t) \right) - \left( a(t), e(t) - v(t) \right) - (e(t), v(t)) \leq (a(t), u(t) - v(t)) - (e(t), v(t)) \leq (a(t), u(t) - v(t)) \Delta(t).
\]

According to Theorem 2.1, \( a(t) \leq L_C \) and \( e(t) \leq L_D \), almost everywhere, thus integration yields (2.7).

\]
It may be observed that the following schemes are of vital importance to many proofs for sweeping processes. We fix $n \in N$ and choose a time discretization

$$0 = t_0^n < t_1^n < \cdots < t_{N-1}^n < t_N^n = T,$$

with $(t_{i+1}^n - t_i^n) \leq \frac{1}{n}$, $0 \leq i \leq N - 1$. \hspace{1cm} (2.8)

For example, we can set $t_i^n = i/n$, but we need not fix the discretization explicitly. The value of $N \in N$ will depend on $u$, and $N \to \infty$, for $n \to \infty$. We define the step approximation $u_i^n : [0, T] \to H$ as follows. Let

$$u_i^n = u_0, \quad u_i^n + \text{proj}(u_{i+1}^n, C(t_{i+1}^n)) \in C(t_{i+1}^n), \quad 0 < i < N - 1. \hspace{1cm} (2.9)$$

This is the "catching up" algorithm, since the approximation $u_i^n$ is made to catch up with the set $C(t_{i+1}^n)$ through projection. Recall that we have to achieve $u(t) \not\in C(t)$ for the solution.

The $u_i$ are defined via linear interpolation

$$u_i(t) = u_i^n + \left( \frac{t - t_i^n}{t_{i+1}^n - t_i^n} \right) (u_{i+1}^n - u_i^n), \quad t \in [t_i^n, t_{i+1}^n]. \hspace{1cm} (2.10)$$

In order to prove existence of a solution, we want to find a subsequence of $(u_i^n)_{n \to \infty}$ that converges to a solution of (1.1) or one of its variants. To this end, we wish to apply Theorem 2.2 and we have to derive the uniform bound in norm and variation in (2.3).

3. The State-Dependent Sweeping Process

In this section we discuss a generalization of the classical sweeping process given by (1.1) where we allow the underlying set $C(t)$ to depend also on the current state $u - u(t)$, so moving set now becomes $C(t, u(t))$. Thus, our new problem is to find a solution $u(t)$ of (3.1) such that

$$-u_t(t) \in \mathcal{N}_{C(t, u(t))}(u(t)) \quad \text{a.e. in } [0, T], \quad u(0) = u_0 \in C(0, u_0). \hspace{1cm} (3.1)$$

Similarly as before a solution $u$ of (3.1) must satisfy $u(t) \in C(t, u(t))$ for $t \in [0, T]$. In order to prove its existence, we need the following property instead of (2.5)

$$d_H(C(t, u), C(t, v)) \leq L_2 |t - s| + L_2 |u - v|, \quad t, s \in [0, T]. \hspace{1cm} (3.2)$$

An important special case of (3.1) is given by the following evolution quasi-variational inequality.

Find $v : [0, T] \to H$ with $v(t) \in \Gamma(v(t))$ such that

$$\langle v'(t) + f(t, u - v(t)), w - v(t) \rangle \geq 0 \quad \text{for all } w \in \Gamma(v(t)), x(0) = u_0 \in \Gamma(u_0), \hspace{1cm} (3.3)$$

where $f : [0, T], H$ is some inhomogeneity, and $\Gamma(v) \subset H$ is a set of constraints.
(3.3) can be written in the form:

\[-c'(t) \in N_{T(t)}(v(t)) + f(t) \text{ a.e. in } [0, t^*], v(0) = v_0 \in \Gamma(v_0).\]

(3.4)

Thus if \( v \) is a solution of (3.4) and if we define \( \eta(t) = v(t) + \int_0^t \int f(s)ds \) and

\[
C(t, u) = \Gamma \left( v - \int_0^t \int f(s)ds \right) + \int_0^t f(s)ds,
\]

(3.5)

then \( v \) is a solution of (3.1) with the initial condition \( \eta_0 = v_0 \in C([0, t^*]). \)

While dealing with (3.3a), we always assume that

\[
\delta_H(\Gamma(\eta), \Gamma(u)) \leq \eta_H - \eta_w \quad \eta, u \in H.
\]

(3.6)

It may be remarked that elliptic and evolution (in particular, parabolic) quasi-variational inequalities have been studied independently by several authors such as Barroso, Cazenave, Lions, Kobayashi, Kunze, Mosco, Ongari, Prigozhin, Rodrigues, Zhou see for more references [1, 3, 9, 11, 33, 32, 34, 35]. Prigozhin [33] has modeled single-phase growth by a parabolic quasi-variational inequality. Kobayashi, Ongari and Zhou have studied algorithms for quasi-variational inequalities. Recently, Carlier, Lions has indicated that parallel algorithms for evolution quasi-variational inequalities could be studied. Kunze and Rodrigues [32] consider a class of quasi-variational inequalities for a second-order elliptic operator and apply it to stationary problems arising in surface diffusion, thermoplasticity, and in electromagnetism with implicit formation threshold. Sekisun and Mezrich [35] have proved two existence theorems, one for evolution quasi-variational inequalities and the other for a time-dependent quasi-variational inequality modeling the quasistatic problem of elastoplasticity with combined kinematic and isotropic hardening.

In general (3.1) may not have a solution. However, if \( L \leq 1 < \gamma \) in (3.2), then (3.1) has a solution. Consequently, the quasi-variational inequality (3.3) is solvable with the restriction \( \gamma \leq 1 < \gamma \). Theorem 3.1 [18] has a solution on \([0, T] \).

Theorem 3.1. Let \( L \leq 1 < \gamma \) and let \( C(t, u) \subset H \) be nonempty, closed, and convex for \( t \in [0, T] \) and \( u \in H \). Assume that

\[
\bigcup_{\alpha \in A} C(t, u) \cap \Pi_H(0) = \emptyset.
\]

(3.7)

is a relatively compact subset of \( H \) for all bounded \( \alpha \subset H \) and all \( R > 0 \). If \( u_0 \in C([0, t^*]), (3.1) \) has a solution on \([0, T] \).

Obviously, the compactness-condition above is always satisfied if \( H \) is finite dimensional.

The proof by time discretization now leads to the implicit discrete equation

\[
e_i^0 = \text{proj}(e_i^0), C(t_i^*, x_i^0). \quad i = 1, 2, 3, \ldots, N.
\]
For this the following lemma plays the key role.

Lemma 3.1 If \( t \in [0, T] \) and \( u \in C_x(t, s) \) for some \( s \in [0, T] \), then there exists \( v \in H \) such that \( v = \text{proj}(u, C(t, s)) \) and \( |v - u| \leq L_1(t - s)/(1 + L_2) \).

The proof of this lemma is based on Schauder's fixed point theorem and an inequality due to Moreau concerning projections, for details see [21].

4. Degenerate Sweeping Processes

Sweeping processes of the following type

\[-u(t) \in N_{\mathcal{R}}(u(t)) \quad \text{a.e. in } [0, T], \quad u(0) = u_0 \in \text{dom}(A), \quad A u_0 \in C(0) \]  

(4.1)

are known as degenerate sweeping process; they may fail to have solutions even in the case where \( A \) is linear, bounded, selfadjoint and satisfies \( (Au, u) \geq 0 \).

For example, let \( H = \mathbb{R}^2 \), \( [0, T] = [0, 1], \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) and \( C(t) = [0, 1] \times \{t, 1\} \) for \( t \in [0, 1] \). (4.1) has no solution in this case with initial condition \( u_0 = (0, 0) \in \text{dom}(A) \). Degenerate sweeping processes have been discussed in references [16, 17, 21] and references therein. Karaz and Monteiro Marques [17] have proved the following theorem:

Theorem 4.1 Let \( A : H \to H \) be linear bounded and selfadjoint such that

\[ (Au, u) \geq \alpha |u|, \quad \alpha > 0. \]  

(4.2)

(2.5) holds for \( u \in C(t) \) and \( A u_0 \in C(0) \). Then (4.1) has a unique solution, which is Lipschitz continuous.

It may be observed that Theorem 2.1 is obtained by choosing \( A = I \) in Theorem 4.1.

5. Variants of Sweeping Processes

The following variant of the sweeping process in (2.1) has been studied by Siddiqui, Marchand and Brokate [36]:

Find \( v : [0, T] \to H \), where \( H \) is a separable Hilbert space such that

\[-u(t) \in N_{\mathcal{R}}(v(t)), \quad \text{a.e. in } [0, T], \quad u(0) = u_0. \]  

(5.1)

Theorem 5.1 Assume that \( t \in [0, T] \subset C(t) \) satisfies (2.5) and \( C(t) \subset H \) is closed and convex for every \( t \in [0, T] \). Moreover, assume that \( C \) is uniformly bounded, that is, there exists \( K > 0 \) such that \( C(t) \subset B_K(0) \) for all \( t \). Then (5.1) has a unique solution which also satisfies (2.4).

Proof. For any \( u \in \mathcal{R} \), set \( b_u = T^* u \) and let \( (v^*_u)_{0 < u < \infty} \) denote the corresponding equidistant partition of \( [0, T] \). We want to define a discrete
solution \((s_i^*)\), \(0 \leq i \leq n\), by
\[
-s_i^* \in N_{C(t_i)} \left( \frac{u_i^* - u_{i-1}^*}{t_i - t_{i-1}} \right), \quad u_0^* = s_0. \tag{5.2}
\]

Introduce (instead of \(u_i^*\)) a new unknown
\[
z = \frac{u_0^* - u_{n+1}^*}{t_0 - t_{n+1}}.
\]

Then (5.2) is equivalent to
\[
-s_i^* - (t_i - t_{i-1})z \in N_{C(t_i)}(z)
\]
which is equivalent to
\[
(I + N_{C(t_i)})z \in \frac{u_0^* - u_{n+1}^*}{t_0 - t_{n+1}}. \tag{5.3}
\]

Because \(N_{C(t_i)}\) is maximal monotone,
\[
\text{Range}(I + N_{C(t_i)}) = H
\]

Therefore (5.3) has a solution \(z \in H\). Since \(z\) belongs to the domain of \(N_{C(t_i)}\), we must have \(z \in C(t_i) \subset B_{\delta_i}(0)\), thus
\[
\left| \frac{u_i^* - u_{i-1}^*}{t_i - t_{i-1}} - z \right| \leq \delta_i, \quad \forall i.
\]

We now define the piecewise linear interpolates
\[
u_n : [0, T] \to H
\]
by
\[
u_n(t) = u_{n-1}^* + (t - t_{n-1}) \frac{u_n^* - u_{n-1}^*}{t_n - t_{n-1}}, \quad t \in (t_{n-1}, t_n).
\]

Since
\[
u_n(t) = \frac{u_n^* - u_{n-1}^*}{t_n - t_{n-1}} + (t - t_{n-1}) \frac{u_n^* - u_{n-1}^*}{t_n - t_{n-1}}, \quad t \in (t_{n-1}, t_n)
\]
we have
\[
|\nu_n(t)| \leq K.
\]

We now perform the passage to the limit. Due to Theorem 2.2, there exists a function \(u : [0, T] \to H\) such that, for a suitable subsequence, \(\nu_n(t) \to u(t)\).
weakly in $H$. On the other hand, the sequence $(u'_n(t))_n$ is bounded in $L^2(0, T; H)$. Therefore, for a suitable subsequence again denoted by $u_n$, we have $u'_n \to u$ weakly in $L^2(0, T; H)$. By passing to the limit in

$$u_n(t) = a_0 + \int_0^t u'_n(s) \, ds,$$

we see that

$$u(t) = a_0 - \int_0^t u(s) \, ds$$

holds for all $t \in [0, T]$, thus $u' = w$ a.e. in $[0, T]$. For the remainder of the convergence argument, we write $u_n$ instead of $u_n$. We have

$$u'_n(t) \in C(t_0) \quad \text{if} \quad t \in (t_0 - h_n, t_0),$$

so

$$\|u'_n(t) - C(t_0)\| \leq d_H(C(t_0), C(t)) \leq K \|u'_n - t\| \leq L h_n, \quad \text{a.e. in } t.$$ 

Because of this estimate, we can apply Lemma 2.1 to the closed convex subset $C = \{r : r \in L^2(0, T; H), r(t) \in C(t) \text{ a.e.}\}$ of the Hilbert space $L^2(0, T; H)$ to conclude that $u' \in C$, thus $u'(t) \in C(t)$ a.e. in $[0, T]$. It remains to prove that

$$\langle u(t), u'(t) - z \rangle > 0 \quad \forall \, z \in C(t) \quad (5.4)$$

holds a.e. in $[0, T]$. Fix $t \in (t_0 - h_n, t_0)$ and $z \in C(t)$. We have for any $z \in H$

$$\langle -u_n(t), u'_n(t) - z \rangle = \langle -u_n(t) + u'_n(t'), u'_n(t') - z \rangle + \langle -u'_n, u'_n(t) - z \rangle.$$ 

Choose $z \in C(t)$ such that $|z - z| < K h_n$, then by (5.2)

$$\langle -u_n(t), u'_n(t) - z \rangle > 0,$$

and

$$\langle -u_n(t), u'_n(t) - z \rangle \geq \langle -u_n(t) + u'_n(t), u'_n(t) - z \rangle - K h_n \|u_n\|_{\infty}.$$ 

Now, for every $r \in L^2(0, T; H)$ with $r(t) \in C(t)$ a.e. we have

$$\left| \frac{\partial}{} \sum_{n=1}^N \int_0^T \langle -u_n(t) + u'_n(t), r(t) \rangle \, dt \right| = 0.$$
so
\[
\int_0^T (-u(t), u'(t) - v(t)) \, dt = \lim_{\alpha \to \infty} \int_0^T (-u_\alpha(t), u'_\alpha(t) - v(t)) \, dt \geq 0
\]
holds for all \( t \) with \( v(t) \in C(t) \) a.e. Passing to (3.4) in the standard manner, we conclude the proof of existence.

To prove uniqueness, let \( u_1, u_2 : [0, T] \to H \) be solutions of (5.1) with initial values \( u_1(0) - u_2(0) = 0 \) and \( u'_1(0) - u'_2(0) = 0 \). Then
\[
(-u_1(t), u_1'(t) - u_2(t)) \leq 0, \quad (-u_2(t), u_2'(t) - u_1(t)) \leq 0,
\]
so
\[
\frac{1}{2} \frac{d}{dt} \|u(t) - u(t)\|^2 \leq 0
\]
and
\[
\|u(t) - u(t)\| \leq \|u_0 - u_0\|.
\]
This implies uniqueness.

Manchanda and Siddiqui have studied the following variant of the state-dependent sweeping process
\[
\begin{align*}
\alpha \in N_C(u(t)(u'(t))) & \quad \text{a.e. in } [0, T] \\
u'(t) = -v(t) & \quad u(0) = u_0 \in C(0, u_0).
\end{align*}
\] (5.5)

**Theorem 5.2.** Let \( C(t, u) \) be a nonempty, convex and closed set \( \subseteq [0, 2] \), \( u \in H \), and \((\cdot, n) : C(t, u) \) satisfies (3.2) for \( 0 \leq L < 1 \) and (3.7) hold. Then (5.5) has a solution.

6. The Sweeping Process Without Convexity

In recent years, some efforts have been made to study Moreau's sweeping process in the setting of nonconvex sets \( C(t) \) (see, for example, [2, 8]). Ren and Wang [2] have proved the following theorem extending Theorem 2.1 for nonconvex subset \( \subseteq \) a finite-dimensional normed space.

**Theorem 6.1.** Let \( C : I \to \mathbb{R}^n \) be a multi-valued that there exists a constant \( L > 0 \) and \( u_0(t) \subseteq C(t) \). \( C(t) \subseteq I \), \( ||x - y|| = 0 \) holds for all \( t, \xi \in I \). Let \( u_0 \subseteq C(t) \). Then there exists an absolutely continuous function \( u : I \to \mathbb{R}^n \) such that
\[
\begin{align*}
u'(t) & \subseteq N_C(t) \subseteq (u(t)) \quad \text{a.e. in } I \quad (6.1) \\
u(t) & \subseteq C(t) \quad \text{for all } t \in I \quad (6.2) \\
u(t) & = u_0 \quad (6.3)
\end{align*}
\]

Theorem 6.1 has been extended to infinite-dimensional spaces by Colombo and Goncharov [8].
7. The Play and Stop Operator

Let us come back to the sweeping process in its original form,

\[ \dot{u}(t) \in N_{C(t)}(u(t)) \]  \hspace{1cm} (7.1) \]

Let us consider the special case of a purely translational motion

\[ C(t) = v(t) - Z \]

where \( Z \) is a fixed closed, convex and nonempty subset of \( H \) and \( v: [0, T^*] \rightarrow H \) is a given function, which we now call the input function. The evolution variational inequality corresponding to (7.1), namely

\[ -(\dot{u}(t), x - u(t)) \leq 0, \quad \forall x \in C(t) \]

can be equivalently written as

\[ \langle \dot{u}(t), v(t) - u(t) - \zeta \rangle \geq 0, \quad \forall \zeta \in Z \]  \hspace{1cm} (7.2) \]

The initial condition must have the form

\[ u(0) = v(0) - z_0, \quad z_0 \in Z \]

If we additionally introduce the function \( z = v - u \), we see that the sweeping process takes on the equivalent form

\[ u(t) - z(t) = v(t), \quad z(0) = z_0 \]  \hspace{1cm} (7.3) \]

\[ z(t) \in Z \quad \forall \zeta \in Z \]  \hspace{1cm} (7.4) \]

The existence and uniqueness theorem for the sweeping process yields for every input function \( v \) and every initial value \( z_0 \), a unique pair of functions \( (u, z) \) which solve (7.3), (7.4). The corresponding solution operators

\[ u = P(v; z_0), \quad z = S(v; z_0) \]  \hspace{1cm} (7.5) \]

are called the play operator and the stop operator, respectively. They constitute basic elements of the mathematical theory of rate independent hysteretic processes; for example, the celebrated Preisach model in ferromagnetism can be written as a nonlinear superposition of a continuous one-parameter family of play operators.

As with the sweeping process, there is a direct geometric interpretation of the play and the stop operator. Let us consider the translational movement defined by

\[ Z(t) = v(t) + Z \]  \hspace{1cm} (7.6) \]

Now the input function \( v \) governs the movement of the convex set \( Z \) which is required to follow \( v \) as \( v(t) \in Z(t) \). Moreover according to (7.4) its velocity vector \( \dot{u}(t) \) lies within the normal cone \( N_Z(\dot{z}(t)) \), where \( \dot{z}(t) = v(t) - u(t) \)
8. Relationship between Variational Inequalities and a Few Open Problems

W. Han, B.D. Reddy, and G.C. Schroeder [12] have studied the following abstract variational problem:

**Problem 8.1**

Find \( w : [0,T] \rightarrow H, \ w(0) = 0 \), such that for almost all \( t \in (0,T) \), \( w(t) \in K \) and

\[
\begin{align*}
\alpha(t, \psi(t)) &+ j(z) - j(\phi(t)) - \langle k(t), \psi(t) \rangle = 0, \\
\psi(t) &\in K, \\
\end{align*}
\]

(8.1)

where \( H \) denotes a Hilbert space, \( K \) a nonempty, closed, convex and convex cone in \( H \); \( \alpha(-, -) \) denotes a real bilinear, symmetric, bounded and \( H \)-elliptic form on \( H \times H; f \in H^{1/2}(a,T; H^*) \) and \( j(\cdot) \) denotes non-negative, convex, positively homogeneous and Lipschitz continuous functional on \( K \) into \( R \).

Siddiqui and Manchanda [35] have studied the following quasi-variational problem:

**Problem 8.2**

Find \( u \in K(u) \cap C, u(0) = 0 \) such that for almost all \( t \in [0,T] \),

\[
\alpha(u(t), \nu - \tilde{\nu}(t)) \geq \langle j(t), \nu - \tilde{\nu}(t) \rangle, \quad \forall \nu \in K(u).
\]

(8.2)

Natural questions are:

(1) What is the relationship between Problem 8.1 and (5.1)?

(2) Is it possible to find a variant of the result of Kunst and Monteiro Marques (5.1) and Theorem 3.1 which will include Problem 8.2 as a special case?

The following classes of variational inequalities are discussed in Duvaut and Lions [9] and Glowinski, Lions and Trélat [11, pp. 454-473].

**Problem 8.3**

It is worthwhile to investigate a class of sweeping process which include these evolution variational inequalities.

Find \( u \in K \):

\[
\begin{align*}
\alpha(u(t), \nu - u^{(t)}(t)) + j'(\nu) - j'(\tilde{u}(t)) &\geq \langle k(t), \nu - \tilde{u}'(t) \rangle, \\
\alpha(u^{(t)}(t), \nu - u(t)) + j(t) - j(\tilde{u}(t)) &\geq \langle k(t), \nu - \tilde{u}'(t) \rangle.
\end{align*}
\]

(8.3)

**Problem 8.4**

Raymond [34] has generalized the Lax-Milgram lemma in the following form:
Theorem 8.1 Let $H$ be a real Hilbert space and $A$ a linear operator on $H$. If
\[
\inf_{x \in H} \{ \|Ax\| : \|x\| = 1 \} > 0.
\] (8.4)
the operator $A$ is continuous and invertible.

In this theorem, the continuity of $A$ has been relaxed in the form of (8.4).

An interesting problem could be to explore the possibility of replacing condition (4.2) in Theorem 4.1 by a weaker condition (8.4).

Problem 8.8 Obtain an analogous result to Theorem 4.1 for the state-dependent sweeping process given by (3.1).

Problem 8.6 Could we prove a result analogous to Theorem 6.1 for the state-dependent sweeping process, that is, to say, could we prove existence and uniqueness of solution of state-dependent sweeping process (3.1) under appropriate conditions?

Problem 8.7 On the lines of Brahideliah [2] and Colombo and Gomme [8] one may try to prove existence and regularity of play and stop operators similar to Theorem 7.1 (relaxing the convexity of the underlying set and Theorem 1.1 in [10]).

Problem 8.8 In recent years, parallel algorithms for evolution variational inequalities have been studied by Lions (see, for example, reference in Siddiqui and Marchand [35]). Proceeding along the lines of Lions one may introduce $N$ Hilbert spaces $H_k$ and a family of linear, bounded operators $r_i \in L(H_k, H_k)$, $i = 1, 2, \ldots, N$. For a given family of Hilbert space $H_k$ such that $H_k = H_{k+1}$, $i = 1, 2, \ldots, N$ and a family of operators $r_i$ such that $r_i \in L(H_k, H_{k+1})$, one may decompose (1.1) into $N$ inclusions $-r_i(\gamma) \in N_i(\gamma)$ (or $r_i(\gamma)$) plus appropriate terms containing $r_i$ and $u_0$, a.e. in $(0,T), \gamma(0) = u_0 \in C((0,1), i = 1, 2, \ldots, N$).

Does this system of inclusions have a unique solution/solutions; whether this solution/solutions converge(s) to the solution of (1.1).

References


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