VARIANTS OF MOREAUS SWEEPING PROCESS

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Abstract: In this paper we prove the existence and uniqueness of two variants of Moreau's sweeping process \(-u'(t) \in NC_0(t(u(t)))\), where in one variant we replace \(u(t)\) by \(u'(t)\) in the right-hand side of the inclusion and in the second variant \(u'(t)\) and \(u(t)\) are respectively replaced by \(u''(t)\) and \(u'(t)\).

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1 Introduction

In the seventies Moreau [14] introduced and studied the evolution problem

\[-u'(t) \in NC_0(t(u(t))) \quad \text{a.e. in } [0,T], \quad u(0) = u_0 \in C(0) \quad (1.1)\]

which describes the motion of a ball inside a ring. Here \(u(t)\) is the position of the ball at time \(t\) and \(C(t)\) is the ring at time \(t\). \(NC_0(t(u(t)))\) denotes the outward normal cone to the set \(C(t)\) at the position \(u(t)\). Thus (1.1) tells us that the velocity \(u'(t)\) of the ball has to point inwards to the ring at almost
every time $t \in [0, T]$. The initial condition $u(0) \in C(0)$ states that the ball is initially contained in the ring. (1.1) is known as the Moreau's sweeping process. This includes evolution variational inequality as a special case. Find $u(t) \in K$ a.e. such that

$$
\langle u'(t), v - u(t) \rangle \geq \langle f, v - u(t) \rangle
$$

(1.2)

for all $v \in K$, $K$ is a subset of a Hilbert space $H$, $u : [0, T] \to H$, $f \in L_2(0, T; H')$.

Several extensions and applications of the Moreau sweeping process in diverse fields [7]-[18], [20]-[22] have been studied. For a lucid introduction of this process along with numerical aspects and applications we, particularly, refer to Moreau [13]. While studying the heat control problem one encounters the following evolution variational inequality:

Find $u = u(x, t)$ such that $u'(t) = \partial u(-t, \xi)/\partial \xi \in H^1(\Omega)$ and

$$
\langle u(t), v - u(t) \rangle + \alpha(u(t), v - u(t)) + j(v) - j(u(t)) \leq \langle f(t), v - u(t) \rangle
$$

(1.3)

where $f(t)$ is convex and lower semicontinuous with values in $(-\infty, +\infty)$ but not identically $+\infty$ (for details see [4, 80-94] and [5, 454-476]). In particular we may consider variational inequality [1-6, 16-19] of the type

Find $u = u(x, t) \in C(t)$ such that $u'(t) \in H'(t)$

$$
\langle u'(t), v - u(t) \rangle \geq 0
$$

(1.4)

and look for existence and uniqueness of solution of a variant of Moreau process, namely

Find $u = u(x, t) \in C(t)$ such that $u'(t) \in C(t)$ and

$$
-u'(t) \in N_C(u'(t))
$$

(1.5)

which includes (1.4) as a special case.

The variational inequality of the type (1.6) is the formulation of the dynamic analogue of the Signorini problem (see [4, 154-160] and [5, 478-483]). Find $u'(t) \in C(t)$ for all $t$ such that

$$
\langle u'\tau(t), \bar{v} - u'(t) \rangle + \alpha(u(t), \bar{v} - u'(t)) + j(\bar{v}) - j(u'(t)) \geq \langle f(t), \bar{v} - u'(t) \rangle
$$

(1.6)

for all $\bar{v} \in C(t)$ and with the initial conditions $u(0) = u_0$, $u'(0) = \alpha_0$.

A natural question is whether the following sweeping process has a unique
solution: Find \( u(t) \in C(t) \) such that \( u'(t) \in C(t) \) a.e. \( t \) and
\[
-u''(t) \in N_{C(t)}(u'(t)), \quad u(0) = u_0, \quad u'(0) = u_1.
\] (1.7)

The main goal of this paper is to study existence and uniqueness of sweeping processes described by (1.5) and (1.7).

2 Notation and Preliminaries

Let \( H \) be a Hilbert space with an inner product \( \langle \cdot, \cdot \rangle \). For a closed convex subset \( C \) of \( H \) the set
\[
N_c(z) = \{ y \in H \mid \langle y, v - z \rangle \leq 0, \forall v \in C \}, \quad z \in C,
\]
denotes the normal cone to \( C \) at \( z \). Let \( d_H(A, B) \) denote the Hausdorff distance between two subsets \( A \) and \( B \) of \( H \) and it is defined as follows
\[
d_H(A, B) = \max_{x \in A} \{ \sup_{z \in B} d(z, A), \sup_{z \in B} d(z, B) \}
\] (2.1)
where \( d(z, A) = \inf \{ ||z - y|| \mid y \in A \} \).

For any Banach space \( X \), we denote by \( C^m([0, T]; X) \) the space of continuous functions \( u : [0, T] \to X \) that have continuous derivatives up to and including those of order \( m \) on \([0, T]\) with the norm
\[
||u||_{C^m(0, T; X)} = \sum_{i=0}^{m} \max_{t \in [0, T]} ||u^{(i)}(t)||_X
\] (2.2)
and by \( L^p(0, T; X) \) for \( 1 \leq p < \infty \) the space of all measurable functions \( u : (0, T) \to X \) for which
\[
||u||_{L^p(0, T; X)} = \left( \int_0^T ||u(t)||_X^p \right)^{1/p} < \infty.
\] (2.3)

The space of measurable functions \( u : (0, T) \to X \) which is essentially bounded and denoted by \( L_{\text{ess}}(0, T; X) \) and this space is endowed with norm
\[
||u|| = \text{ess sup}_{0 \leq t \leq T} ||u(t)||_X
\] (2.4)

Some properties of those spaces are listed in Theorem 2.1 [23].

**Theorem 2.1** Let \( m \) be a nonnegative integer and \( 1 \leq p < \infty \). Let \( X \) be a Banach space.
a) $C^m([0,T];X)$ with the norm (2.2) is a Banach space.

b) $L_p(0,T;X)$ is a Banach space if we identify functions that are equal almost everywhere in $(0,T)$.

c) If $X$ is a Hilbert space with inner product $(\cdot, \cdot)_X$ then $L_2(0,T;X)$ is also a Hilbert space with the inner product

$$
(u,v)_{L_2(0,T;X)} = \int_0^T \langle u(t), v(t) \rangle_X dt.
$$

(2.3)

The topological dual of a Banach space $X$ is defined by $X^*$ and the operation of an element $x^* \in X^*$ on an element $x \in X$ is represented by $(x^*, x)$. If $X$ is separable the $L_1(0,T;X^*)$ is separable and $(L_1(0,T;X))^* = L_\infty(0,T;X^*)$. If $X$ is a Hilbert space then $(L_2(0,T;X))^* = L_2(0,T;X^*)$. For a Hilbert space $H$, we define by $W^{1,2}(0,T;H)$ the space of functions $u \in L_2(0,T,H)$ such that $u' \in L_2(0,T;H)$, equipped with the norm

$$
\|u\|_{W^{1,2}(0,T;H)} = \|u\|_{L_2(0,T;H)}^2 + \|u'\|_{L_2(0,T;H)}^2
$$

(2.6)

where $u'$ denotes the generalized derivative of $f$ on $(0,T)$. A function $w = u^{(n)}$ is the generalized derivative of the function $u$ on $(0,T)$ if and only if

$$
\int_0^T \phi^{(n)}(t) u(t) \, dt = \lim_{h \to 0} \frac{1}{h} \int_0^T \phi(t) (u(t+h) - u(t)) \, dt
$$

(2.7)

for all $\phi \in C^\infty_c(0,T)$ - the space of infinitely differentiable functions having compact support. The integrals in (2.7) exist if $u \in L_1(0,T,H)$. The generalized derivative is unique, if the function $u \cdot [0,T] \to H$ is continuous and the derivative

$$
u(t) = \lim_{h \to 0} \frac{u(t+h) - u(t)}{h}
$$

(2.8)

exist for all $t \in [0,T]$ as a limiting value in $H$; and $u' : [0,T] \to H$ is also continuous then $u$ is the generalized derivative of $u$ on $(0,T)$. Moreover, if $u \in L_1(0,T;H)$ then $u' \in L_1(0,T;H^*)$. The following results [23] are needed in our subsequent discussion.

**Theorem 2.2** ([23], p.421) Let $H$ be a Hilbert space and let $u \cdot [0,T] \to H$ be Lipschitz continuous, that is

$$
\|u(t) - u(s)\| \leq L|t - s| \quad \text{for all } t, s \in [0,T]
$$

(2.9)

and fixed $L \geq 0$. Then
a) For almost all \( t \in [0, T] \), the function \( u \) has a derivative,

\[
u'(t) = \lim_{h \to 0} \frac{u(t + h) - u(t)}{h}
\]

and

\[
u(t) = u(0) + \int_0^t \nu'(s) \, ds \quad \text{for all } t \in [0, T]
\]

b) For almost all \( t \in [0, T] \)

\[\|u'(t)\| \leq L\]

and \( u' \) is the generalized derivative of \( u \) on \((0, T)\).

An operator \( A : H \to H^* \) is called monotone if

\[
(Au - Av, u - v) \geq 0 \quad \text{for all } u, v \in H
\]

(2.10)

A is called strongly monotone if there is a constant \( \beta > 0 \) such that

\[
(Au - Av, u - v) \geq \beta \|u - v\|^2
\]

(2.11)

A is maximal monotone if and only if

\[R(A + I) = H\]

For this characterization which was also proposed by Minty and other related results see [23, chapter 32].

It may be observed that if \( X = \mathbb{R} \) then \( u' \) for \( u : X = \mathbb{R} \to X^* = \mathbb{R} \) is strongly monotone if \( u \) is \( C^1 \) and \( u'(t) > c \) for all \( t \in \mathbb{R} \) and fixed \( c > 0 \). \( u' \) is strongly monotone if \( u \) is \( C^1 \) and satisfies \( u'(t) - u'(s) \geq c(t - s) \) for all \( t, s \in \mathbb{R} \) and \( c > 0 \).

Let

\[
D_1(u) = u' \quad \text{dom}(D_1) = \{ u \in W^{1,2}(0, T; H) \mid u(0) = 0 \} \subseteq H
\]

\[
D_2(u) = u' \quad \text{dom}(D_2) = \{ u \in W^{1,2}(0, T; H) \mid u(0) = u(T) \} \subseteq H
\]

Then \( D_1 : \text{dom}(D_1) \to H^* \) and \( D_2 : \text{dom}(D_2) \to H^* \) are maximal monotone operators.

A moving set valued map \( t \mapsto C(t) \) is called Lipschitz continuous if

\[
d_C(C(t), C(s)) \leq L|t - s|, \quad t, s \in [0, T]
\]

(2.12)

for some constant \( L > 0 \). Our aim is to prove that for a Lipschitz continuous moving set \( C(t) \) there exists a unique solution to (1.3). By a solution of (1.5) we mean a function \( u : [0, T] \to H \) such that
a) \( u(0) = u_0 \)

b) \( u(t) \in C(t) \) for almost every \( t \in [0,T] \)

c) \( u'(t) \in C(t) \) for almost every \( t \in (0,T) \)

d) \(-u'(t) \in N_{C(t)}(u(t))\) for almost every \( t \in [0,T] \)

The following discretization process is needed for the proof of the solution of the sweeping process. We fix \( n \in N \) and choose a time discretization

\[
0 = \eta_1^n < \eta_2^n < \ldots < \eta_{m-1}^n < \eta_m^n = T
\]

(2.13)

with \( \eta_{i+1}^n - \eta_i^n \leq \frac{T}{n}, 0 \leq i \leq m - 1 \). We may set \( \eta_0^n = 0 \), but we need not fix the discretization explicitly. The value of \( m \) will depend on \( n \) and \( m \to \infty \) for \( n \to \infty \). We define the step approximation \( u^n : [0,T] \to H \) as follows. Let

\[
u^n_i = u_i, \quad u_{i+1}^n = u_i^n + \text{proj}(0, C(\eta_i^{n+1})) \in C(\eta_i^{n+1})
\]

(2.14)

\( 0 \leq i \leq m - 1 \). The \( u_n \) are defined via linear interpolation

\[
u_n(t) = u_i^n + \frac{t - \eta_i^n}{\eta_{i+1}^n - \eta_i^n}(u_{i+1}^n - u_i^n), \quad t \in [\eta_i^n, \eta_{i+1}^n].
\]

(2.15)

For \( x \in H \) an element \( y \) of \( C \) is called the projection of \( x \) on \( C \subset H \) (\( C \) is closed and convex) written as

\[
y = \text{proj}(x, C) \text{ if } \|x - y\| = \min_{z \in C} \|x - z\|.
\]

(2.16)

Equivalently \( y = \text{proj}(x, C) \) if

\[
(y - x, y - z) \leq 0 \text{ for all } z \in C.
\]

(2.17)

For our discussion we assume that \( 0 \in C(t) \) and \( C(t) \) is a cone and \( u'(t) \in C(t) \) whenever \( u'(t) \) exists and \( u(t) \in C(t) \).

3 Existence Results and related Lemmas

Theorem 3.1 Let \( t \to C(t) \) be Lipschitz continuous, that is, satisfy (2.12) and \( C(t) \subset H \) be nonempty, closed and convex for every \( t \in [0,T] \). Let \( u_0 = u(0), u_0' = u'(0) \) belong to \( C(0) \). Then there exists a unique solution \( u : [0,T] \to H \) of (1.5) which is Lipschitz continuous. In particular, \( u \in L_{\infty}(0,T; H) \) and \( u' \in L_{\infty}(0,T; H) \).
Theorem 3.2 Let $t \to C(t)$ be Lipschitz continuous, that is, satisfy (2.12) and $C(t) \subset H$ be nonempty, closed and convex for every $t \in [0, T]$. Let $u_0 = u(0), u_2 = u''(0)$ belong to $C(0)$. Then there exists a unique solution $u : [0, T] \to H$ of (1.7) which is Lipschitz continuous. In particular, $u \in L^0(0, T; H), u' \in L^0(0, T, H)$ and $u'' \in L^2(0, T, H)$.

Lemma 3.1 ([13], p.10) Let $H$ be a Hilbert space and $(\alpha_n)$ be a sequence of functions $\alpha_n : [0, T] \to H$ that is bounded uniformly in norm and variation, that is,
\[ \|\alpha_n(t)\| \leq M_1, \quad \alpha_n \in N, \quad t \in [0, T] \text{ and } \var(\alpha_n) \leq M_2, \quad \alpha_n \in N \]
(3.1)
for some constants $M_1, M_2 > 0$ independently of $n \in N$ and $t \in [0, T]$. Then there exists a subsequence $\{\alpha_{n_k}\}$ and a function $\alpha : [0, T] \to H$ such that $\var(\alpha) \leq M_2$ and $\alpha_{n_k}(t) \to \alpha(t)$ weakly in $H$ for all $t \in [0, T]$, that is,
\[ \alpha_{n_k}(t, z) \to (u(t), z) \quad \text{for all } z \in H \]
(3.2)
as $k \to \infty$.

Lemma 3.2 ([10] or [23], p.258)

a) Let $u_n \to u$ weakly in $H$. The
\[ \|u\| \leq \liminf_{n \to \infty} \|u_n\| \]
(3.3)
holds.

b) If $u_n \in C + B_C(0)$ for some closed convex $C \subset H$ and some sequence $\epsilon_n \to 0$, then $u \in C$.

Lemma 3.3 (Rockafellar, R.T. see [10]) Let $\{u_n\}$ be a sequence of functions $u_n : [0, T] \to H$ such that $u_n \to u$, in the weak* topology of $L^0([0, T], H)$, that is,
\[ \int_0^T (u_n(t), \phi(t)) \, dt \to \int_0^T (u(t), \phi(t)) \, dt \quad \text{as } n \to \infty \]
(3.4)
for all $\phi \in L^2([0, T], H)$. Suppose that for each $t \in [0, T]$ the set $C(t) \subset H$ is nonempty, closed and convex such that (2.12) is satisfied. Let
\[ \Phi(u) = \int_0^T \delta'(u(t), C(t)) \, dt \]
(3.5)
for $u \in L^0(0, T; H)$, where $\delta'(z, C) = \sup\{\langle z, c \rangle \mid c \in C\}$ for $z \in H$. Then $\Phi$ is lower semi-continuous, that is,
\[ \Phi(u) \leq \liminf_{n \to \infty} \Phi(u_n). \]
Lemma 3.4 ([10]) Let \( u : [0, T] \to H \) be a continuous function that is differentiable at almost every point \( t \in (0, T) \). Then

a) \( \frac{7}{6} \int_0^T (u'(t), u(t)) \, dt = \frac{1}{2} \| u(T) \|^2 - \frac{1}{2} \| u(0) \|^2 \)

b) \( \frac{3}{2} \int_0^T (\| u'(t) \|^2) = (u'(t), u'(t)) = \| u'(t) \|^2 \).

4 Proof of Theorem 3.1

Step 1. First of all we show that if \( u \) is a weak limit of \( u_n \) given by (2.15) then \( u \in L_{w}(0, T; H) \), that is, \( \| u(t) \| \leq M \) for almost every \( t \in [0, T] \). It can be seen that

\[
\| u_{n+1} - u_n \| \leq d_B (C_1(t_n), C_1(t_{n+1})) \leq L | t_n - t_{n+1} | \tag{4.1}
\]

where we have used discretization in Section 2, (2.14) and (2.12). If \( u_n \) is defined by (2.15) then

\[
\text{var}(u_n) = \sum_{n=1}^{m-1} \| u_{n+1} - u_n \| = \sum_{n=1}^{m-1} \| u_{n+1} - u_n \| 
\]

\[
\leq L \sum_{n=1}^{m-1} (t_{n+1} - t_n) = LT = M_1,
\]

\[
\| u_{n+1} \| \leq \| u_n \| + L (t_{n+1} - t_n)
\]

and

\[
\| u_n(t) \| \leq \| u_0 \| + L (\| u_{n+1} - u_n \|)
\]

\[
\leq \| u_0 \| + L (t_{n+1} - t_n)
\]

\[
\leq \| u_0 \| + LT = M_1 \tag{4.2}
\]

for \( t \in [t_n, t_{n+1}] \). Consequently the desired result \( \| u_n(t) \| \leq M_1 \) holds for \( t \in [0, T] \) as the above relation is true for all \( n \in N \) and \( t \in [0, T] \). Since \( \{ u_n(t) \} \) is a bounded sequence in Hilbert space \( H \) we can extract a subsequence still denoted by \( u_n(t) \) which converges weakly in \( H \) say \( u_n(t) \rightharpoonup u(t) \) weakly for all \( t \in [0, T] \) (Lemma 3.1).

Step 2. \( t \to u(t) \) is Lipschitz continuous.

Let \( u_n(0) = u_0 \) then weak limit of \( u_n(0) \) is \( u(0) = u_0 \). For \( t \in [t_n, t_{n+1}] \) and
\[ s \in [t_i^n, t_{i+1}^n] \text{ for some } 0 \leq j, i \leq m - 1, \text{ (without loss of generality we can assume } i \leq j, \text{ that is, } s_i^j \leq s^j \) from (2.12) and (4.1) we obtain

\[
\|u_s(t) - u_s(s)\| \leq \|u_s(t) - u_s(s_i^j)\| + \sum_{k=0}^{j-1} \|u_s(t_{k+1}^j) - u_s(s_i^j)\|
+ \|u_s(t_{j+1}) - u_s(s)\| \\
\leq \frac{t - s_i^j}{t_{j+1}^n - t_i^n} \|u_s(t_{j+1}) - u_s(s_i^j)\| + \sum_{k=0}^{j-1} \|u_s(t_{k+1}^j) - u_s(t_k^j)\|
+ \frac{t_i^j - s}{t_{j+1}^n - t_i^n} \|u_s(t_{j+1}^n) - u_s(t_i^j)\|
\leq \sum_{k=0}^{j} \|u_s(t_{k+1}) - u_s(t_k)\| \leq \sum_{k=0}^{j} \|u_s(t_{k+1}^n) - u_s(t_k^j)\| = L(t_{j+1}^n - t_i^j)
\]

or

\[
\|u_s(t) - u_s(s)\| \leq L[t - s_i^j] + |s - t_i^j| + |t_{j+1}^n - t_i^j| \leq L\left(\|t - s_i^j\| + \frac{2}{n}\right). \tag{4.3}
\]

By (4.3) and Lemma 3.2(a) we get

\[
\|u(t) - u(s)\| \leq \liminf_{n \to \infty} \|u_s(t) - u_s(s)\| \leq L|t - s|
\]
as weak limit of \((u_s(t) - u_s(s)) = u(t) - u(s)\). Therefore \(u\) is Lipschitz continuous and by Theorem 2.2(a), \(u(t)\) exists for almost every \(t\) and \(\|u(t)\| \leq L\) by Theorem 2.2(b). Hence \(u' \in L_u(0, T; H)\). Clearly \(u(0) = u_0\).

Step 3. To show that \(u(t) \in C(t)\).

By (2.13) and (2.15) we have

\[
\|u(t) - u(s)\| \leq \liminf_{n \to \infty} \|u_s(t) - u_s(s)\| \leq L|t - s|
\]
for \(t \in [t_i^n, t_{i+1}].\) Hence by (2.15), (2.16) and (2.12), for \(t \in [t_i^n, t_{i+1}],\)

\[
\|u(t)\| \leq C(t_s^i) + B(t_{i+1}^-)(0) \leq C(t) + B(t_{i+1}^-)(0) + B(t_{i+1}^-)(0) \leq C(t) + B(t_{i+1}^-)(0).
\tag{4.4}
\]

(2.13) has been used in the last step. It is clear that (4.4) holds for all \(n \in N\) and \(t \in [0, T],\) and so Lemma 3.2 yields \(u(t) \in C(t)\) for all \(t \in [0, T].\)
By (2.14) and (2.17) we have

$$ (u_n^* - u_0^*, u_n^* - u_0^* - e) \leq 0, \quad v \in C(t_n^*). $$  \hspace{1cm} (4.5)

From (2.15), (4.1) and (2.13) we obtain

$$ ||u_n(t) - u_n^*|| = \frac{t_n^* - t}{t_n^* - t_0^*} ||u_n^* - u_0^*|| \leq L(t_n^* - t) \leq \frac{L}{n}, $$ \hspace{1cm} (4.6)

for \( t \in [t_0^*, t_n^*]. \) Since by (2.12),

$$ C(t) \subset C(t_n^*) + \tilde{B}_{L(t_n^* - t_0^*)}(0) \subset C(t_n^*), $$

for \( t \in [t_0^*, t_n^*], \) we find from (4.6) and (4.1) that

$$ (u_n^* - u_0^*, u_n^* - v) = (u_n^* - u_0^*, u_n^* - u_0^* - e) + (u_n^* - u_0^*, (u_n^* - u_0^*) - (u_n^* - u_0^*)) + [v - v] \leq ||u_n^* - u_0^*|| \frac{L + L}{n} \leq \frac{M}{n}, $$ \hspace{1cm} (4.7)

for \( t \in (t_0^*, t_n^*], \) where \( M = \max_{t \in (t_0^*, t_n^*)} ||u_n^* - u_0^*|| \) and hence by (4.7) we get

$$ (u_n^*(t), u_n^*(t) - v) \leq \frac{M}{n}, \quad t \in (t_0^*, t_n^*), $$ \hspace{1cm} (4.8)

\( v \in C(t). \) The estimate (4.1) also shows that

$$ ||u_n^*(t)|| \leq L, \quad t \neq t_n^*, $$

hence

$$ ||u_n^*(t)||_{L^n(0,T; H)} \leq L, \quad n \in N. $$

Since \( L^0(0, T; H) \) is the dual space of \( L^0(0, T; H), \) it is a consequence of the Banach-Alaoglu theorem that we may extract a further subsequence, again indexed by \( n \) (see for example [10] or [23], p. 269), such that \( u_n^* \to u^* \) for some \( u^* \in L^0(0, T; H) \) the consequence being in the weak star topology on \( L^0(0, T; H). \) This means that for all \( \phi \in L^1(0, T; H) \)

$$ \int_0^T (u_n^*(t), \phi(t)) \, dt \to \int_0^T (u^*(t), \phi(t)) \, dt \text{ as } n \to \infty. $$

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According to the differentiability of $u_n$, 
\[ u_n(t) = u_0 + \int_0^t u'_n(s) \, ds, \quad t \in [0, T]. \]

It can be seen that 
\[ u_n(t) = u_0 + \int_0^t u_s(s) \, ds, \quad t \in [0, T]. \]

This again shows $u : [0, T] \rightarrow H$ is differentiable for almost every point $t \in (0, T)$, and moreover $u'(t) = u_n(t)$ for almost every $t \in (0, T)$. In particular 
\[ -u'_n \rightarrow -u' \quad \text{in the weak star topology on} \quad L_{w0}(0, T; H). \]

By lemma 3.3 this gives 
\[ \int_0^T \delta'(-u'(t), C(t)) \, dt \leq \lim_{n \to \infty} \inf \int_0^T \delta'(-u'_n(t), C(t)) \, dt \quad (4.9) \]
(for a definition of $\delta'(\cdot, \cdot)$ see Lemma 3.3). It is clear that 
\[ \int_0^T \langle u'(t), u'(t) \rangle \, dt \leq \lim_{n \to \infty} \inf \int_0^T \langle u'_n(t), u'_n(t) \rangle \, dt \quad (4.10) \]

Taking supremum w.r.t. $v$ in (4.6) and integrating over $[0, T]$ we find that 
\[ \int_0^T [\delta'(-u'_n(t), C(t)) + \langle u'_n(t), u'_n(t) \rangle] \, dt \leq \frac{MT}{n} \quad (4.11) \]
for $n \in \mathbb{N}$. Using a well known property of the limit inferior of a sequence (4.9), (4.10) and (4.11) we get 
\[ \int_0^T [\delta'(-u'(t), C(t)) + \langle u'(t), u'(t) \rangle] \, dt \leq 0 \quad (4.12) \]

We have shown in Step 3 that $u(t) \in C(t), t \in [0, T]$, and so $u'(t) \in C(t)$. By the definition of $\delta'(\cdot, \cdot)$ we get 
\[ \delta'(-u'(t), C(t)) + \langle u'(t), u'(t) \rangle = 0 \]
for almost every $t \in (0, T)$. Thus for any $v \in C(t)$ 
\[ \langle -u'(t), u(t) \rangle = \delta'(-u'(t), C(t)) \geq \langle -u'(t), v \rangle \]
or 
\[ \langle -u'(t), v - u(t) \rangle \leq 0. \]
Hence $u(t)$ is a solution of (1.5)

**Step 5. Uniqueness of solution.**

Let $u_1$ and $u_2$ be two solutions of (1.5) then

$$(-u_1'(t), v - u_1(t)) \leq 0$$  \hspace{1cm} (4.13)

and

$$(-u_2'(t), v - u_2(t)) \leq 0.$$  \hspace{1cm} (4.14)

Put $v = u_2'(t)$ and $v = u_1(t)$ respectively in (4.13) and (4.14) then we get

$$(-u_1'(t), u_2'(t) - u_1(t)) \leq 0$$  \hspace{1cm} (4.15)

$$(-u_2'(t), u_1(t) - u_2'(t)) \leq 0.$$  \hspace{1cm} (4.16)

From (4.15) and (4.16) we get

$$u_1'(t) - u_2'(t), u_1'(t) - u_2'(t) \leq 0$$

or

$$||w'(t)||^2 = 0$$ \hspace{1cm} (4.17)

where $w(t) = u_1'(t) - u_2'(t)$. From (4.17) we get

$$\int_0^t |w'(\xi)|^2 d\xi = 0$$ as $w(0) = 0$.

Be Lemma 3.4

$$\int_0^1 \frac{1}{2} \frac{d}{d\xi} (|w'(\xi)|^2) = 0$$

or $w(\xi) = 0, \forall \xi \in (0, T)$ or $u_1(\xi) = u_2(\xi), \forall \xi \in (0, T)$.

### 5 Proof of Theorem 3.2

Let $u(t) = \phi(t)$ and $u(t) = \phi'(t)$. Then $\phi'(t) \in N_{CIP}(\phi(t))$ for almost every $t \in (0, T)$ holds by Theorem 2 [10] (Theorem 2.1 [13, p. 141] or Moreau [14]) provided $\phi(t) \in C(t)$. $t \rightarrow \phi(t)$ is Lipschitz continuous with constant $L$. In particular $|\phi'(t)| = |u(t)| \leq L$ for almost every $t \in (0, T)$ and so $u''(t) \in L^1(0, T; H')$. Let $\phi(t)$ and $\phi_1(t)$ be two solutions of (1.7) then

$$(-\phi_1'(t), v - \phi_1(t)) \leq 0$$ \hspace{1cm} (5.1)
\[ (-\phi_1(t), v - \phi_2(t)) \leq 0. \] (5.2)

By (5.1) and (5.2) we get
\[ (\phi_1'(t) - \phi_2'(t), \phi_1(t) - \phi_2(t)) \leq 0. \] (5.3)

By Lemma 3.4 we have
\[ \frac{1}{2} \frac{d}{dt} \| \phi_1(t) - \phi_2(t) \|^2 = \| \phi_1'(t) - \phi_2'(t), \phi_1(t) - \phi_2(t) \| \leq 0 \]
almost everywhere in \((0, T)\). Integration yields
\[ \| \phi_1(t) - \phi_2(t) \|^2 \leq \| \phi_1(0) - \phi_2(0) \|^2 = \| \phi_1^0 - \phi_2^0 \|^2 \]
for all \( t \in [0, T] \). In particular, if \( \phi_1^0 = \phi_2^0 \) then the solution is unique.

6 Relationship with Degenerate Sweeping Processes

Kunze and Monteiro Marques [8] have proved the following theorems.

**Theorem 6.1** Let \( A : \text{dom}(A) \to 2^H \) be a maximal and strongly monotone operator and for any \( t \in [0, T] \), \( C(t) \neq \emptyset \subset H \) be closed and convex set and \( t \to C(t) \) be Lipschitz continuous. If in addition the following conditions are satisfied

a) \( C(0) \) is bounded or there exists a function \( M : [0, \infty) \to [0, \infty) \) which maps bounded sets such that
\[ \| Ax \| = \sup \{ \| y \| : y \in Ax \} \leq M(\| x \|) \quad \text{for } x \in \text{dom}(A), \]

b) \( \text{dom}(A) \cap B_r(0) \) is relatively compact for every \( r > 0 \) or \( C(t) \cap B_r(0) \) is compact for every \( t \in [0, T] \) and \( r > 0 \).

Then there exists a Lipschitz continuous function \( u : [0, T] \to H \), \( u(t) \in \text{dom}(A) \) a.e., such that for every \( w_0 \in \text{dom}(A) \) with \( Aw_0 \cap C(0) \neq \emptyset \)
\[ u(t) \in Aw(t) \cap C(t) \quad \text{a.e.} \]

and
\[ -u(t) \in N_{C(t)}(v(t)) \quad \text{a.e. in } [0, T] \] (6.1)
Theorem 6.2 Let \( A : H \rightarrow H \) be linear, bounded and self adjoint such that 
\[ \langle Ax, x \rangle \geq \beta \|x\|^2 \] 
for \( x \in H \). If \( x \rightarrow C(t) \) is Lipschitz continuous where 
\( t \in [0, T] \), \( C(t) \subseteq H \) is closed and convex and \( Au \in C(0) \), then (6.1) has a 
unique solution which is Lipschitz continuous.

It may be observed that in some special cases Theorem 3.1 and Theorem 3.2 
can be derived from Theorem 6.1 and Theorem 6.2. For example, if \( A \) is as 
\( D_0 \) or \( D_1 \) defined in Section 2, \( H = R \) and \( u \) is of \( C_\alpha \) class with \( u'(t) > c \) 
for all \( t \in R \) and fixed \( c \) then Theorem 6.1 reduces to Theorem 3.1 provided 
\( C(0) \) is bounded.

If we choose \( A = u' \) in Theorem 6.2 then \( u' \) satisfies the condition \( \langle u', u \rangle \geq \beta \|u\|^2 \) 
is linear and self adjoint. However \( u' \) is bounded only almost everywhere 
and so Theorem 3.1 cannot be obtained as a special case of Theorem 6.2.

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