4-7 General 3-D Stress

In general for a 3-D stress, there are 3 principal stresses: \( \sigma_1 \), \( \sigma_2 \) and \( \sigma_3 \). A 3-D stress element with the 3 principal stresses are shown below:

![3-D Stress Element Diagram]

The Mohr’s circle for the above stress element would look like this:

![Mohr's Circle Diagram]

The extreme shear stresses are:

\[
\tau_{1/2} = \frac{(\sigma_1 - \sigma_2)}{2}, \quad \tau_{1/3} = \frac{(\sigma_1 - \sigma_3)}{2} \quad \text{and} \quad \tau_{2/3} = \frac{(\sigma_2 - \sigma_3)}{2}.
\]

The maximum shear stress is:

\[
\tau_{\text{max}} = \tau_{1/3}
\]

Even for a 2-D plane stress analysis, one can say that there are 3 principal stresses. \( \sigma_1 \) and \( \sigma_2 \) are found using the equations given before, but \( \sigma_3 = 0 \). There are 3 scenarios, which are:

1) \( \sigma_1, \sigma_2 > 0, \sigma_3 = 0 \);  
2) \( \sigma_1, \sigma_2 < 0, \sigma_3 = 0 \) and  
3) \( \sigma_1 > 0, \sigma_2 < 0, \sigma_3 = 0 \).
Assuming $\sigma_1 > \sigma_2$, for the first case: $\tau_{\text{max}} = \tau_{1/3} = (\sigma_1 - \sigma_3) / 2 = \sigma_1 / 2$, for the second case: $\tau_{\text{max}} = \tau_{2/3} = (\sigma_2 - \sigma_3) / 2 = \sigma_2 / 2$ and for the third case: $\tau_{\text{max}} = \tau_{1/2} = (\sigma_1 - \sigma_2) / 2$.

4-8, 9 Elastic Strain and Stress

For a bar subject to a tensile axial load of $F$, the bar elongates with an amount of $\delta$, then the axial elastic strain $\varepsilon$ is defined as

$$\varepsilon = \delta / L$$

where $L$ is the original length of the bar. The axial stress $\sigma$ is simply

$$\sigma = F / A$$

where $A$ is the cross-sectional area of the bar. The Hooke’s Law is given as

$$\sigma = E \varepsilon$$

where $E$ is the modulus elasticity of the material of the bar, which is given as 207 GPa for the carbon steel. Refer to Table A-5, p. 963, in the Appendix A of the textbook, for the physical constants of materials. From the above relations, one can obtain the elongation of the bar to be

$$\delta = (FL)/(AE).$$

Also, the shear stress $\tau$ for a bar subjected to a direct shear force of $F$ (scissor action)

$$\tau = F / A$$

for which the Hooke’s Law is expressed as

$$\tau = G \gamma$$

where $\gamma$ is the shear angle and $G$ is the modulus of rigidity related to $E$ as

$$E = 2G(1+\nu)$$

in which the symbol $\nu$ stands for the Poisson’s ratio defined as

$$\nu = -\text{Lateral Strain/Axial Strain}.$$ 

The Poisson’s ratio can be taken as 0.3 for the carbon steel, again refer to Table A-5 in the textbook.

For the general elastic stress-strain relations (Hooke’s Laws) for the uniaxial (1-D), biaxial (2-D) and triaxial (3-D) stress cases, refer to Table 4-2, p. 124, in the textbook.
4-10 Normal Stresses for Beams in Bending

A cut beam subject to a positive bending moment $M$ is shown below:

![Diagram of a beam with normal stresses indicated]

The normal stress at a distance $y$ from the neutral axis is given by:

$$\sigma_x = -(M/I)y$$

where $I$ is the area moment of inertia or the second moment of area about the $z$ axis. The stress at the top surface is compression whose value is $\sigma_x = -(M/I)c$ and at the bottom surface the stress is tension given by $\sigma_x = (M/I)c$. The moment of inertia for a solid round cross-section is: $I=\pi d^4/64$ and therefore the normal stress is given by: $\sigma_x = 32M/(\pi d^4)$. For a rectangular cross-section of width $b$ and height $h$, $I = bh^3/12$ and $\sigma_x = 6M/(bh^3)$.

**Note:** Review Examples 4-5 and 4-6 in textbook, pages 127-130.

4-12 Shear Stresses for Beams in Bending

The shear force $V$ for a beam in bending causes shear stresses, which are maximum at the neutral axis, and zero at the top and bottom surfaces. The cut beam is shown below:

![Diagram of a beam with shear stress indicated]

For a rectangular cross-section, the shear stress on a stress element at a distance $y$ from the neutral axis is given by:
\[ \tau_{xy} = 3V(1-y^2/c^2)/(2A) \]

Hence, it is zero at the top and bottom surfaces where \( y = \pm c \); and it is maximum at the neutral axis with a value of \( \tau_{xy} = 3V/(2A) \). For a solid round cross-section,

\[ \tau_{xy} = 4V(1-y^2/c^2)/(3A) \]

And again, it is zero at the top and bottom surfaces where \( y = \pm c \); and it is maximum at the neutral axis with a value of \( \tau_{xy} = 4V/(3A) \). For different cross-sections, see Table 4-3, p.136, in the textbook.

### 4-13 Torsion

The twisting torque \( T \) on a round bar or shaft causes angular displacements and shear stresses as shown below:

![Torsion Diagram](image)

The angular displacement or angle of twist \( \theta \) is given as:

\[ \theta = TL/(GJ) \]

where \( L \) is the length of the bar and \( J \) is the polar moment of inertia or second polar moment of area about the centroid of the cross-section. For a solid round cross-section with a diameter of \( d \), \( J \) is given as \( J = \pi d^4/32 \). The shear stress caused by the torque \( T \) on a round bar is given by:

\[ \tau = T\rho/J \]

where \( \rho \) is the radial distance of the stress element from the center of the cross-section. This shear stress is maximum at the outside surface, which is at a distance of \( \rho = r \) from the center. This maximum stress is

\[ \tau = Tr/J = 16T/(\pi d^3) \]

for a solid round cross-section at the outside surface. The maximum shear stress for a rectangular cross-section of \( b \times c \), where \( b \) is the longer side, i.e. \( b > c \):

\[ \tau = T(3 + 1.8c/b)/(bc^2) \]

**Note:** Review Examples 4-7 (p.133), 4-8 (p.139) and 4-9 (p. 141) in the textbook.
4-14 Stress Concentration

The elastic stress across the cross-section of a machine element is uniform in the case of a bar in tension, or linear as in the case of a beam in bending. Many times, machine elements are required to have holes, notches, grooves etc. due to various reasons. For example, a shaft may be drilled a hole because of mounting a gear onto it. Such discontinuities disturb the stress distribution in machine elements and cause stress concentrations. See, for example, the following plate with a hole of diameter $d$ subjected to tension:

At the section of A-A, the nominal stress $\sigma_o$ is

$$\sigma_o = \frac{F}{t(b-d)}$$

where $t$ is the thickness of the plate. But, due to the stress concentration at the edge of the hole, this stress rises to $\sigma_{\text{max}}$, which is given as

$$\sigma_{\text{max}} = K_t \sigma_o$$

where $K_t$ is called the theoretical stress concentration factor for normal stress and $K_t \geq 1$. We have a similar scenario for the case of shear stress where:

$$\tau_{\text{max}} = K_{ts} \tau_o$$

where $K_{ts}$ is the theoretical stress concentration factor for shear stress and again $K_{ts} \geq 1$. The values of $K_t$ and $K_{ts}$ depend on the type of loading, i.e. tension, bending, torsion, and also on the geometry. The values for several cases are given in the Appendix of the textbook, Table A-15, pages 982-988.