**Question 1:** (13 Points)

a. Find the dual of the following linear programming problem (*Don't solve*)

(5 points)

Minimize \( z = 2x_1 - 3x_2 - 5x_3 \)

Subject to \( x_1 - 2x_2 \geq -1 \)

\( 2x_1 + x_2 - x_3 \leq 5 \)

\( x_2 + x_3 \leq 4 \)

\( x_1, x_2, x_3 \geq 0 \)

**Solution:** Change constraints to:

\[ x_1 - 2x_2 \geq -1 \]

\[ -2x_1 - x_2 + x_3 \geq -5 \]

\[ -x_2 - x_3 \geq -4 \]

\( x_1, x_2, x_3 \geq 0 \)

The dual:

Maximize \( W = -y_1 - 5y_2 - 4y_3 \)

Subject to:

\[ y_1 - 2y_2 \leq 2 \]

\[ -2y_1 - y_2 - y_3 \leq -3 \]

\[ y_2 - y_3 \leq -5 \]

\( y_1, y_2, y_3 \geq 0 \)

b. At what nominal rate of interest compounded semiannually, will money tripled in 10 years. (3 points)

**Solution.** Let \( r \) be the semiannual rate

\[ n = (10)(2) = 20 \]

\[ S = P(1 + r)^n \]

\[ 3P = P(1 + r)^{20} \Rightarrow (1 + r)^{20} = 3 \]

\[ 1 + r = \sqrt[20]{3} \]

\[ r = \sqrt[20]{3} - 1 = 0.0565 \]

\[ i. \text{ The nominal rate} = 2r \]

\[ = 2(0.0565) \]

\[ = 0.113 \]

or \( r = 11.3\% \)
c. If a man has a choice of investing a sum of money at 9% compounded annually or 8.8% compounded semiannually, which one he has to choose? Why? (3 points)

I. For annual rate
\[ r_{e} = (1 + r)^4 - 1 = (1 + 0.09)^1 - 1 = 0.09 = 9\% \] 

II. For semiannual interest: \( n = 2 \)
\[ r_{e} = \left(1 + \frac{0.088}{2}\right)^2 - 1 \]
\[ = 0.089936 = 8.994\% \]

So, the man must choose to invest his money annually at 9%.

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d. If $7500 is invested for four years with an interest rate of 5 1/2% compounded continuously, then find the compounded amount and compounded interest.
(2 points)

Solution: \( P = 7500, r = 0.055, n = 4 = t \)

The compounded amount = \( S = Pe^{rt} \)
\[ = 7500 e^{0.055(4)} \]
\[ = 7500 e^{0.22} \]
\[ = 9345.58 \]

The compounded interest = \( S - P \)
\[ = 9345.58 - 7500 \]
\[ = 1845.58 \]
Question 2: (10 Points)

a. A debt of $5500 due in six years from now and $4500 due in ten years from now is to be repaid by a payment of $3000 in two years and $3000 in seven years and final payment at the end of eight years. If the interest rate is 5% compounded semiannually, how much should the final payment? (6 points)

Solution: Let $x$ be the final payment. \( r = \frac{r}{2} = 0.025 \)

\[
\begin{align*}
0 & \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \\
\text{3000} & \quad \text{3000} & \quad x & \quad 5500 \quad (1 + 0.025)^2 + x = 5500 \quad (1 + 0.025)^2 + 4500 \quad (1 + 0.025)^{-4} \\
4034.59 & \quad 3151.845 & \quad x = 6070.991 & \quad 4034.59 + 3151.845 + x = 6070.991 + 4034.59 + 3151.845
\end{align*}
\]

\[
x = 5500(1 + 0.025)^2 + 4500(1 + 0.025)^{-4} - 3000(1 + 0.025)^{-2} - 3000(1 + 0.025)^{-12}
\]

\[
x = 10,147.75 - 7186.545
\]

\[
x = $2,961.21
\]

b. An initial investment of $25,000 in a business guarantees the following cash flows.

<table>
<thead>
<tr>
<th>Year</th>
<th>Cash Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$8,000</td>
</tr>
<tr>
<td>4</td>
<td>$10,000</td>
</tr>
<tr>
<td>6</td>
<td>$11,000</td>
</tr>
</tbody>
</table>

If the interest rate is 4% compounded semiannually, determine whether the investment is profitable or not? Why? (4 points)

Solution: \( r = \frac{0.04}{2} = 0.02 \)

\[
\begin{align*}
\text{The net present value} & = NPV = 8000 \quad (1 + 0.02)^{-6} + 10,000 \quad (1 + 0.02)^{-8} + 11,000 \quad (1 + 0.02)^{-12} - 25,000 \\
& = 3103.771 + 8534.904 + 8643.425 - 25000 \\
& = 243312.1 - 25000 \\
& = -687.9 < 0
\end{align*}
\]

The NPV < 0, then it is not profitable.
Question 3: (7 Points)

(a) Find the present value of an annuity of $250 due at the beginning of each year for three years, and $550 due thereafter at the beginning of each year for four years. If the interest rate is 5% compounded annually? (3 points)

\[ \text{Solution: Method 1:} \quad r = 0.05 \]

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
250 & 250 & 250 & 550 & 550 & 550 & 550 & 550 \\
\end{array}
\]

\[
The \text{ present \ values \ of \ all \ payments} = 250(1+0.05)^6 + 250(1+0.05)^5 + 250(1+0.05)^4 + 550(1+0.05)^3 + 550(1+0.05)^2 + 550(1+0.05) + 550(1+0.05)^6 \\
\]

\[
= 250 + 237.3095 + 226.0577 \quad \{2\}
\]

\[
= $2,483.81 \quad \{0\}
\]

\[
\text{Method 2:} \quad \text{ROR} \quad 3
\]

\[
The \text{ present \ values \ of \ all \ payments} = 550(1+0.05)^3 - 300(1+0.3)^3 = 550(1+0.05)^3 - 300(1+0.3)^3 \\
\]

\[
= 550(1+0.0755692) - 300(1+1.85941) \quad \{0\}
\]

\[
= $2,483.81 \quad \{0\}
\]

(b) In 12 years a $42,000 bus will have a salvage value 20% of its cost. A new bus at that time is expected to sell for $48,000. In order to provide funds for the difference between the replacement cost and the salvage value, a sinking fund is set up into which equal payments are placed at the end of each year. If the fund earns 6% compounded annually, how much should each payment be? (4 points)

\[
\text{Solution: Time} = n = 12 \quad r = 0.06 \quad S = ?
\]

\[
\text{Salvage Value} = 0.20(42,000) = 8,400 \quad \{0\}
\]

\[
The \text{ amount needed after 12 years} = 48,000 - 8,400 = 39,600 \quad \{0\}
\]

\[
S = R \cdot \frac{1}{r} \Rightarrow R = \frac{S}{\frac{1}{r}} = \frac{39,600}{16.869941} \approx $2,347.37 \quad \{0\}
\]

\[
R = \frac{S}{\frac{1}{r}} \text{ where } \frac{1}{r} = (1+r)^{12} = (1.06)^{12} \quad \{0\}
\]

\[
= 16.869942
\]

\[
\therefore \quad R = \frac{39,600}{16.869942} = $2,347.37.
\]
Question 4: (6 Points)

a. Suppose a company wants to make six-digit phone lines, for which the first digit to the left is not zero. How many even phone lines does the company have? (2 points)

Solution:

\[
\begin{array}{cccccc}
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
9 & 10 & 10 & 10 & 10 & 5 \\
\text{(Any number 0, 2, 4, 6, 8)} \\
\end{array}
\]

The number of even phone lines = \(9(10)(10)(10)(10)(5)\)

\[= 450,000\]

b. In how many ways can 8 persons can be distributed to three rooms in a hotel where two rooms with 3 beds, and the third one with 2 beds? (2 points)

Solution:

\[n = 8, \quad n_1 = 3, \quad n_2 = 3, \quad n_3 = 2\]

The number of ways = \[\frac{8!}{3! 3! 2!}\]

\[= 560\]

c. In a 10-question examination, each question is graded right or wrong. The student gets A grade if he answered at least 8 questions, in how many ways can a student get A grade? (2 points)

Solution.

The number of ways = \[10C_8 + 10C_9 + 10C_{10}\]

\[= \frac{10!}{8! 2!} + \frac{10!}{9! 1!} + \frac{10!}{10! 0!}\]

\[= 45 + 10 + 1 = 56\]
Question 5: (5 Points)
An experiment consists of flipping a coin and then flipping it a second time if a head occurs. If a tail occurs on the first flip, then a die is tossed once. (Use: H for head, T for tail)
   a. Write the sample space. (1 point)

\[ S = \{ HH, HT, T1, T2, T3, T4, T5, T6 \} \]

b. Write the elements of the following events: (1 point each)
   \( E \): the number appeared on the die is at least two,
   \( F \): a head occur or the number is odd.

\[ E = \{ T2, T3, T4, T5, T6 \} \]
\[ F = \{ HH, HT, T1, T3, T5 \} \]


c. Do the events \( E \) and \( F \) are mutually exclusive? Why? (2 points)

\[ ENF = \{ T3, T5 \} \]
\[ ENF \neq \emptyset \Rightarrow E \text{ and } F \text{ are not mutually exclusive} \]
Question 6: (9 Points)

A survey was made to study the students' opinion about evening classes. The results are summarized in the following table:

<table>
<thead>
<tr>
<th>Opinion Level</th>
<th>Highly agree</th>
<th>Agree</th>
<th>Disagree</th>
<th>Highly disagree</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Freshman</td>
<td>45</td>
<td>60</td>
<td>135</td>
<td>65</td>
<td>305</td>
</tr>
<tr>
<td>Sophomore</td>
<td>28</td>
<td>15</td>
<td>55</td>
<td>25</td>
<td>123</td>
</tr>
<tr>
<td>Junior</td>
<td>17</td>
<td>10</td>
<td>30</td>
<td>15</td>
<td>72</td>
</tr>
<tr>
<td>Total</td>
<td>90</td>
<td>85</td>
<td>220</td>
<td>105</td>
<td>500</td>
</tr>
</tbody>
</table>

If one student is selected at random, find the following:

a. What is the probability that he student is a junior or disagrees? (3 points)

Let \( J \): The student is junior \( D \): The student disagrees.

\[
p(J \cup D) = p(J) + p(D) - p(J \cap D) = \frac{131}{250} \]

b. What are the odds of the event \( E \): the student is freshman and highly agrees? (3 points)

\[
p(E) = \frac{45}{500} = \frac{9}{100} = 0.09 \]

\[
p(E') = 1 - p(E) = 1 - 0.09 = 0.91 \]

The odds \( \frac{p(E)}{p(E')} = \frac{0.09}{0.91} = \frac{9}{91} \approx 0.0990 \)

C. Two boxes A and B, if A contains three red and four green balls, B contains five red and seven green balls. One ball is selected at random from each box.

I. Write the sample space. (1 point)

\[ A = \begin{pmatrix} R & G \\ 3 & 4 \end{pmatrix} \quad B = \begin{pmatrix} R & G \\ 5 & 7 \end{pmatrix} \]

\[ S = \{ RR, RG, GR, GG \} \]

II. What is the probability of getting one red ball and one green ball? (2 points)

Let \( E \) = \{ one red ball and one green ball \}

\[
p(E) = p(RG) + p(GR) = \frac{3}{7} \cdot \frac{4}{12} + \frac{4}{7} \cdot \frac{5}{12} = \frac{1}{4} + \frac{5}{21} = \frac{41}{84} \]

\[ = 0.488\]