**SOLUTIONS**

**Question 1. (10-Points)**

Solve the following linear programming problem **geometrically.**

Minimize $Z = 3x + 4y$

Subject to:

- $3x - 4y \leq 12$
  - $y \geq \frac{3}{4}x - 3$

- $x + 2y \geq 4$
  - $2y \geq -x + 4 \Rightarrow y \geq -\frac{1}{2}x + 2$

- $x \geq 1$
  - $x \geq 1$

- $y \geq 0$

$y = \frac{3}{4}x - 3 \quad \frac{x | 4}{y | 0} \quad | \frac{8}{3}$

$y = -\frac{1}{2}x + 2 \quad \frac{x | 0}{y | 2} \quad | \frac{1}{0}$

$A = (4, 0) \quad \boxed{1 \text{ pt}}$

$B \rightarrow$ The intersection point between $x = 1 \neq -\frac{1}{2}x + 2$

$x = 1$, $y = -\frac{1}{2}(1) + 2 = \frac{3}{2} + 2 = \frac{3}{2}$

$\therefore B = (1, \frac{3}{2}) \quad \boxed{1 \text{ pt}}$

$Z(A) = Z(4, 0) = 3(4) + 4(0) = 12 + 0 = 12 \quad \boxed{1 \text{ pt}}$

$Z(B) = Z(1, \frac{3}{2}) = 3(1) + 4(\frac{3}{2}) = 3 + 6 = 9 \quad \boxed{1 \text{ pt}}$

$Z$ has a minimum value at $B$ where $x = 1$, $y = \frac{3}{2}$ and the minimum value is $Z = 9 \boxed{2 \text{ pts}}$
Question 2. (10-Points)
A company manufactures three products X, Y and Z. Each product requires the use of time on machines A, B as given in the following table:

<table>
<thead>
<tr>
<th>Product</th>
<th>Machine A</th>
<th>Machine B</th>
<th>Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Product X</td>
<td>1 hr</td>
<td>1 hr</td>
<td>$10</td>
</tr>
<tr>
<td>Product Y</td>
<td>2 hr</td>
<td>1 hr</td>
<td>$15</td>
</tr>
<tr>
<td>Product Z</td>
<td>2 hr</td>
<td>2 hr</td>
<td>$22</td>
</tr>
</tbody>
</table>

The number of hours per week that A and B are available for production are 40 and 34 hours, respectively. The profit per unit on X, Y and Z is $10, $15 and $22, respectively. What should be the weekly production order if maximum profit is to be obtained using the \textbf{simplex method}?

Let \(x_1, x_2, x_3\) be the number of units produced weekly from X, Y and Z respectively.

Let \(P\). The profit obtained.

We want to: \(\text{Maximize } P = 10x_1 + 15x_2 + 22x_3\) \(\geq \) \(\text{subject to:}\)

\[
\begin{align*}
    x_1 + 2x_2 + 2x_3 & \leq 40 \\
    x_1 + x_2 + 2x_3 & \leq 34 \\
    x_1, x_2, x_3 & \geq 0
\end{align*}
\]

\[
\begin{align*}
    x_1 + 2x_2 + 2x_3 + s_1 &= 40 \\
    x_1 + x_2 + 2x_3 + s_2 &= 34 \\
    -10x_1 - 15x_2 - 22x_3 - P &= 0
\end{align*}
\]

\[
\begin{array}{cccccc}
    & x_1 & x_2 & x_3 & s_1 & s_2 & P \\
  s_1 & 1 & 2 & 2 & 1 & 0 & 0 \\
  s_2 & 1 & 1 & 2 & 0 & 1 & 0 \\
  b & 40 & 34 & 0 & 1 & 0 & 1 \\
\end{array}
\]

\[
\begin{array}{cccccc}
    & x_1 & x_2 & x_3 & s_1 & s_2 & P \\
  1R_1 & 1 & 2 & 2 & 1 & 0 & 0 \\
  22R_3 + R_1 & 0 & 1 & 0 & -10 & 0 & 17 \\
  x_3 = 0 & & 0 & 1 & 0 & 1 & 0 \\
  P = & -10 & -15 & -22 & 0 & 0 & 1 \\
\end{array}
\]

\[
\begin{array}{cccccc}
    & x_1 & x_2 & x_3 & s_1 & s_2 & P \\
  2 R_2 + R_1 & 1 & 0 & 1 & -10 & 0 & 6 \\
  17 R_3 & 0 & 1 & 0 & -1 & 0 & 6 \\
  x_3 & 0 & 0 & 1 & -1 & 0 & 14 \\
  P & 1 & 0 & 0 & 4 & 7 & 1 \\
\end{array}
\]

The maximum profit is \(P = \$398\) when \(x_1 = 0, x_2 = 6, x_3 = 14\).
Question 3. (8-Points)
Consider the following graph which shows the feasible region for a linear programming problem:

Let the intersection points be A, B, C, and D as appear on the graph:

\[ A = (4, 0) , \quad B = (3, 2) , \quad C = (0, 4) , \quad D = (0, 10) \]

- \[ Z(A) = Z(4, 0) = 18(4) + 9(0) = 72 + 0 = 72 \] \[ \begin{aligned} &\text{pl} \end{aligned} \]
- \[ Z(B) = Z(3, 2) = 18(3) + 9(2) = 54 + 18 = 72 \] \[ \begin{aligned} &\text{pl} \end{aligned} \]
- \[ Z(C) = Z(0, 4) = 18(0) + 9(4) = 0 + 36 = 36 \] \[ \begin{aligned} &\text{pl} \end{aligned} \]
- \[ Z(D) = Z(0, 10) = 18(0) + 9(10) = 0 + 90 = 90 \] \[ \begin{aligned} &\text{pl} \end{aligned} \]

\[ Z \] has a maximum value at A, B, and C, so it has maximum value at all points \((x, y)\) on the line segment \(A-B\), where

\[ x = (1-t)4 + t3 = 4 - 4t + 3t = 4 - t \]
\[ y = (1-t)10 + t2 = 10 - 10t + 2t = 2t \quad , \quad 0 \leq t \leq 1 \]

\[ \text{OPT} \]
\[ x = (1-t)4 + 4t = 3 - 3t + 4t = 3 + t \]
\[ y = (1-t)2 + 2t = 2 - 2t + 2t = 2 \quad , \quad 0 \leq t \leq 1 \]
Question 4. (7 + 3 = 10 Points)

(a) Write the dual for the following linear programming problem (DO NOT SOLVE)

Minimize \( Z = x_1 + 8x_2 + 5x_3 \)
Subject to: \( x_1 + x_2 + x_3 \geq 8 \)
\( x_1 - 2x_2 - x_3 \leq -2 \)
\( x_1, x_2, x_3 \geq 0 \)

The dual is:

\[
\begin{align*}
\text{Maximize:} \quad W &= 8y_1 + 2y_2 \\
\text{Subject to:} \quad y_1 - y_2 &\leq 1 \\
& \quad y_1 + 2y_2 \leq 8 \\
& \quad y_1 + y_2 \leq 5 \\
& \quad y_1, y_2 \geq 0
\end{align*}
\]

(b) Consider the following linear programming problem:

Minimize \( Z = x_1 + 2x_2 \Rightarrow Z = 2x_1 + 2x_2 \).

Subject to: \( x_1 + x_2 \geq 80 \),
\( 3x_1 + 2x_2 \geq 160 \),
\( 5x_1 + 2x_2 \geq 200 \),
\( x_1, x_2 \geq 0 \)

Suppose after using the simplex method to solve the dual for the above problem the last simplex tableau is given by:

| \( y_1 \) | \( y_2 \) | \( y_3 \) | \( s_1 \) | \( s_2 \) | \( W \) | \( b \) \\
|--------|--------|--------|--------|--------|--------|--------|
| \( y_2 \) | 0 | 4 | 8 | 2 | -1 | 0 | 2 \\
| \( y_1 \) | 4 | 0 | -4 | -2 | 3 | 0 | 2 \\
| \( W \) | 0 | 0 | 40 | 40 | 20 | 1 | 120 |

Find the solution for the objective function \( Z \).

The solution for \( Z \) is: \( x_1 = 40 \), \( x_2 = 20 \), \( Z = 120 \)

(3 points)
Question 5. \((4 + 2 = 6\text{-Points})\)

Over a six-year period, an original principal of 8000 SR is accumulated to 12867.5 SR in an account in which interest was compounded quarterly. Find:

a. The nominal rate of interest

\[
P = 8000, \quad S = 12867.5, \quad 6 \text{ yrs (quarterly)}, \quad r^* = ?
\]

Let \(r\) be the quarterly rate of interest
\[
6 = (6)(4) = 24 \quad \text{(1 pt)}
\]

\[
S = P(1+r)^n \Rightarrow 12867.5 = 8000(1+r)^{24}
\]

\[
\frac{12867.5}{8000} = (1+r)^{24} \quad \text{(2 pt)}
\]

\[
24 \sqrt[24]{\frac{12867.5}{8000}} = 1+r \Rightarrow 24 \sqrt[24]{1.6084535} = 1+r
\]

\[
1.02 = 1+r \Rightarrow r = 1.02 - 1 = 0.02 \quad \text{(3 pt)}
\]

\[\therefore \text{The nominal rate } = 4r = 4(0.02) = 0.08 = 8\% \quad \text{(4 pt)}\]

b. The compounded interest

\[
\text{The compounded interest } = S - P
\]

\[
= 12867.5 - 8000 \quad \text{(3 pt)}
\]

\[
= 4867.5 \text{ SR} \quad \text{(4 pt)}
\]
**SOLUTIONS**

Question 6. (3 + 3 = 6-Points)

a. A trust fund is being set up so that at the end of 15 years there will be 100,000 SR. If interest is compounded continuously at an annual rate of 6.5%, how much money (In SR) should be paid into the fund initially?

\[ t = 15, \quad S = 100,000 \text{ SR} , \quad P = ? , \quad r = 6.5\% \]

\[
P = \frac{S}{e^{rt}} \\
= \frac{100,000}{e^{0.065(15)}} \\
= \frac{100,000}{e^{0.975}} \\
= 37,419.23 \text{ SR} \quad \boxed{1 \text{ pt}}
\]

b. If interest is compounded continuously at an annual rate of 7%, how many years would it take for a principal P to triple (Becomes three times)? Give your answer to the nearest year.

\[ r = 7\% , \quad t = ? , \quad S = 3P \]

\[
S = Pe^{rt} \Rightarrow 3P = Pe^{0.07t} \quad \boxed{1 \text{ pt}}
\]

\[
3 = e^{0.07t} \quad \text{Take ln}
\]

\[
\ln 3 = \ln e^{0.07t} \Rightarrow 0.07t = \ln 3 \quad \boxed{1 \text{ pt}}
\]

\[
\therefore t = \frac{\ln 3}{0.07} \approx 15.6945 \text{ yr}, \quad \boxed{1 \text{ pt}}
\]

\[ \approx 16 \text{ years} \]