THREE PARAMETERS DAMAGE MODEL FOR CONCRETE

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Abstract

An effective compliance matrix $\bar{C}$ is proposed to model the behaviour of concrete based on phenomenological evidences and physical insight. Three parameters $\alpha$, $\beta$ and $\gamma$ were introduced in the effective compliance matrix $\bar{C}$. $\alpha$ and $\beta$ are introduced to model the different behaviour of concrete in tension and compression while the third parameter $\gamma$ was introduced to account for volumetric change. The predictive capability of the proposed elasto-damage model for uniaxial and multiaxial stress path was investigated for uniaxial compression, biaxial compression and uniaxial tension. The predicted results correlate well with the available experimental data. The simulative capability of the model to capture the phenomenological behaviour of volumetric dilatation is reflected. Also, the ability of the model to predict multiaxial response of concrete is reflected through biaxial strength interaction envelope which compares very well with the experimental results.

1 Introduction

In recent years considerable research has been focused on modelling of mechanical behaviour of concrete. The mechanical behaviour of concrete is very complicated and the possible variations in material characteristics have not been modelled effectively under various theoretical frameworks such as non-linear elasticity [1–3], rate-independent plasticity [4–6], endochronic theory [7, 8] and plastic-fracturing theory [9, 10]. Typical trends in concrete behaviour include: stiffness degradation, strain softening, volumetric dilatation, different behaviour in tension and compression, and gain in strength under increasing confinement.

More recently, the theory of continuum damage mechanics (CDM) originally proposed by Kachanov [11] has been used extensively to model the progressive degradation of the mechanical properties of materials caused by microcracking. Concrete contains numerous microcracks, even before the application of the external loads. Under applied loading, the initiation of new microcracks and the growth of existing microcracks contribute to the observed nonlinear behaviour in concrete, ultimately causing the failure. The existence of microcracks and their propagation cause what is termed as “damage” to the concrete.

The effective stress-concept introduced by Kachanov [11] has been applied to concrete by Krajcinovic [12], Loland [13] and Mazars [14]. Scalar damage variable used by them was not sufficient to model the highly oriented cracking occurring in concrete under loading. Vectorial/tensorial damage variables were used to model this crack-induced anisotropy by Krajcinovic and Fonseka [15], Sauris and Shah [16], Kachanov [17], Cordebois and Sidoroff [18] and Ortiz [19]. The damage effect tensor $\bar{M}$ was proposed and used to cast the continuum damage theory for the general case of anisotropic damage in a consistent mathematical and mechanical framework by Chow and Wang [20, 21]. However, application of the work of Chow and Wang was limited to metals. The concept of bounding surface, first applied to metals by Dafalias and Popov [22], was applied to concrete by Fardis et al. [23], Sauris et al. [24], Voyiadis and Abu-Lebedeh [25], Abu-Lebedeh and Voyiadis [26], Yazdani and Schreyer [27] and Yazdani and Karnawat [28]. More recently, the concepts of damage effect tensor and bounding surface were used successfully to predict most of the essential features of normal and high strength concrete except volumetric dilatation by Khan et al. [29].

The present study is aimed towards developing a constitutive model for normal and high strength concrete able to capture all essential features of concrete including volumetric dilatation. This is achieved by proposing effective compliance matrix $\bar{C}$ based on phenomenological evidences and physical insight. Different behaviour of concrete in tension and compression is taken into account by introducing two parameters $\alpha$ and $\beta$, while the effect of volumetric dilatation is controlled by a third parameter $\gamma$. The constitutive relations and damage growth are derived using the approach presented by the authors in their previous work [29]. The proposed mode is simple, captures the constitutive behaviour of concrete and can be directly implemented into a general-purpose finite element code with relative ease.
2 Theoretical Preliminaries

2.1 Effective compliance matrix

Based on physical insight, the following compliance matrix \( C \) is postulated

\[
\begin{align*}
C_{11} &= \frac{1}{E_o} \frac{(1 - \beta \omega)^2}{(1 - \alpha \omega_1)^2(1 - \beta \omega_2)^2(1 - \beta \omega_3)^2} \\
C_{22} &= \frac{1}{E_o} \frac{(1 - \beta \omega)^2}{(1 - \alpha \omega_2)^2(1 - \beta \omega_1)^2(1 - \beta \omega_3)^2} \\
C_{33} &= \frac{1}{E_o} \frac{(1 - \beta \omega)^2}{(1 - \alpha \omega_3)^2(1 - \beta \omega_1)^2(1 - \beta \omega_2)^2} \\
C_{12} &= -\frac{1}{E_o} \frac{1}{1 - \gamma \omega_1} \\
C_{13} &= -\frac{1}{E_o} \frac{1}{1 - \gamma \omega_2} \\
C_{23} &= -\frac{1}{E_o} \frac{1}{1 - \gamma \omega_3}
\end{align*}
\]

(1)

in which the thermodynamic constraint requirement

\[
E_i \nu_{ij} = E_j \nu_{ij}
\]

(2)

has been insured.

2.2 Bounding surface

In order to construct a rational model accounting for damage growth, concepts are borrowed from incremental theory of plasticity in general and the bounding surface plasticity model in particular as introduced by Dafalias and Popov [22]. Plasticity bounding surface model as proposed by Dafalias, requires definition of multiple surfaces in stress space. However, the fundamental surfaces in the present work are best described in strain-energy release space, as proposed by Suaris et al. [24]

\[
f = \left(R_i R_i\right)^{1/2} - R_c / b = 0
\]

(3)

\[
F = \left(\overline{R}_i \overline{R}_i\right)^{1/2} - R_c = 0
\]

(4)

\[
f_o = \left(R_i R_i\right)^{1/2} - R_o = 0
\]

(5)

with the mapping parameter \( b \) ranging from an initial value of \( \infty \) to a limiting value of 1 on growth of loading surface to coalesce with bounding surface. \( R_c \), critical strain energy release rate, is a parameter of the model and is calibrated to the standard uniaxial compression test, and is suggested to be 1.29. \( R_o \) defines the initiation of microcracking which occurs at about 40% of the peak stress as indicated by experimental results, and it varies with the compressive strength of concrete as:

\[
R_o = \sqrt{2} \beta (0.4 f'_c)^2 / E_o
\]

(9)

Damage is hypothesized to accumulate at levels of strain energy release rate resulting in the loading surface \( f \) traversing the limit fracture surface \( f_o \) and rupture in the damage sense is said to occur when \( f'_c \) grows large enough to coalesce with the bounding surface \( F \) fixed in the \( R_i \) space.

2.3 Damage evolution

The damage growth is determined from the loading surface, \( f \), as

\[
d\omega_t = d\lambda \frac{\partial f}{\partial R_i}
\]

(10)

With \( k = R_c / b \), equation of loading becomes

\[
f(R_i, k) = \left(R_i R_i\right)^{1/2} - k (\overline{\omega}_p) = 0
\]

(11)

where \( \overline{\omega}_p \), the norm of accumulated damage is defined as,
\[ d\bar{\sigma}_p = C (d\omega, d\omega) \]

Consistency condition \( df = 0 \)
\[
\frac{\partial f}{\partial R_i} dR_i + \frac{\partial f}{\partial k} dk = 0
\]

From (6)
\[
dR_i = \frac{\partial R_i}{\partial \sigma_k} d\sigma_k + \frac{\partial R_i}{\partial \omega_j} d\omega_j
\]

Substituting in (13)
\[
\frac{\partial f}{\partial R_i} \left( \frac{\partial R_i}{\partial \sigma_k} d\sigma_k + \frac{\partial R_i}{\partial \omega_j} d\omega_j \right) + \frac{\partial f}{\partial k} \frac{\partial \sigma}{\partial \sigma_p} d\bar{\sigma}_p = 0
\]

Substituting (10) in (12)
\[
d\bar{\sigma}_p = C \left[ d\lambda \left( \frac{\partial f}{\partial R_i} \frac{\partial f}{\partial R_i} \right)^{1/2} \right]
\]

It can be shown that \( \left( \frac{\partial f}{\partial R_i} \frac{\partial f}{\partial R_i} \right)^{1/2} = 1 \) and for \( C = 1 \)
\[
d\bar{\sigma}_p = d\lambda
\]

Using (10), (11) and (17) in (15) and solving for \( d\lambda \) yields
\[
d\lambda = \frac{\frac{\partial f}{\partial k} \frac{\partial R_i}{\partial \sigma_k} d\sigma_k}{\frac{\partial \sigma}{\partial \sigma_p} - \frac{\partial f}{\partial R_i} \frac{\partial R_i}{\partial \omega_j} dR_j}
\]

Introducing \( H = \frac{\partial k}{\partial \sigma_p} \) = damage modulus, it can be measured experimentally in a uniaxial compression test and the same form is assumed for more general stress paths.

In the present work, \( H \) is expressed as a function of the distance between the loading and the bounding surface, given by
\[
H = \frac{D\delta}{\langle \delta_{in} - \delta \rangle}
\]

where \( D = 2.65 \) is a constant and \( < > \) are Macaulay brackets that set the quantity within it to zero if the value is negative. The normalized distance \( \delta \) between the loading and bounding surfaces is given by
\[
\delta = 1 - \frac{1}{b}
\]

\( \delta = \delta_{in} \) corresponds to \( R_o \) when the loading surface first crosses the limit fracture surface [24].

3 Incremental Stress-Strain Relations

3.1 Elasto-damage compliance matrix

Total form of stress-strain law can be expressed as
\[
\epsilon_i = C_{ij} (\omega_r) \sigma_j
\]

The incremental form of Equation (21) is given by
\[
de_i = C_{ij} \sigma_j + \sigma_j \frac{\partial C_{ij}}{\partial \omega_k} d\omega_k
\]

or
\[
de_i = \tilde{C}_{ij} \sigma_j + \sigma_j \frac{\partial \tilde{C}_{ij}}{\partial \omega_k} d\omega_k
\]

or
\[
de_i = C_{ij}^{ed} \sigma_j
\]

where \( d\lambda \) is defined in Equation (18), and \( C_{ij}^{ed} \) is the elasto-damage compliance matrix as defined in Equation (24). Equation (25) is useful in a stress control testing.

3.2 Elasto-damage stiffness matrix

The constitutive equation for strain control can be expressed as
\[
\sigma_i = D_{ij} (\omega_r) \epsilon_j
\]

where
\[
D = \tilde{C}^{-1}
\]

Also we need
\[
R_i = -\rho \frac{\partial W}{\partial \epsilon_r} (\epsilon_r, \omega_r)
\]

where,
\[
\rho W = \frac{1}{2} [\epsilon] \rho [D] [\epsilon]
\]

the complementary strain energy density.
\[ dR_i = \frac{\partial R_i}{\partial \epsilon_j} d\epsilon_j + \frac{\partial R_i}{\partial \omega_j} d\omega_j \]  
(30)

and

\[ d\lambda = \frac{\frac{\partial f}{\partial R_k} \frac{\partial R_k}{\partial \epsilon_j} d\epsilon_j}{H - \frac{\partial f}{\partial R_k} \frac{\partial \lambda}{\partial \omega_j} \frac{\partial \lambda}{\partial \omega_j}} \]  
(31)

The incremental form of Equation (26) is given by

\[ d\sigma = D_{ij} d\epsilon_j + \epsilon_j \left[ \frac{\partial D_{ij}}{\partial \omega_k} d\lambda \frac{\partial f}{\partial R_k} \right] d\epsilon_j \]  
(32)

\[ d\sigma = \left( D_{ij} + \epsilon_i \frac{\partial D_{ij}}{\partial \omega_k} \frac{\partial f}{\partial R_k} \frac{\partial \lambda}{\partial \epsilon_j} \right) d\epsilon_j \]  
(33)

or

\[ d\sigma = D_{ij}^{ed} d\epsilon_j \]  
(34)

where \( d\lambda \) is defined in Equation (31), and \( D_{ij}^{ed} \) is the elasto-damage stiffness matrix as defined explicitly in Equation (33). Equation (34) is useful in strain control testing.

### 4 Application of Proposed Elasto-Damage Model

#### 4.1 Uniaxial compression

The strain energy density \( \rho \Lambda \) for uniaxial compression is given by

\[ \rho \Lambda = \frac{1}{2} \begin{bmatrix} \sigma \end{bmatrix} \begin{bmatrix} \epsilon_j \\ \omega_j \end{bmatrix} \]  
(35)

Substituting for the compliance \( \begin{bmatrix} C \end{bmatrix} \), and setting \( \alpha = 0 \) (as it is used primarily as a parameter for matching peak strengths in tension testing), one obtains

\[ \rho \Lambda = \frac{\sigma^2 \sigma}{2E_o (1-\beta \omega_1)^2 (1-\beta \omega_3)^2} \]  
(36)

The thermodynamic relation

\[ \epsilon_j = \frac{\partial (\rho \Lambda)}{\partial \sigma_j} \]  
(37)

yields

\[ \sigma = \epsilon_j E_o \frac{(1-\beta \omega_2)^2 (1-\beta \omega_3)^2}{(1-\beta \omega_1)^2} \]  
(38)

Substitution of (38) in (36) yields

\[ \rho W = \frac{\epsilon_j^2 E_o}{2} \frac{(1-\beta \omega_2)^2 (1-\beta \omega_3)^2}{(1-\beta \omega_1)^2} \]  
(39)

Using the relationship

\[ R_i = -\frac{\partial (W)}{\partial \omega_i} \]  
(40)

one obtains

\[ R_1 = -\frac{\beta \epsilon_j^2 E_o (1-\beta \omega_2)^2 (1-\beta \omega_3)^2}{(1-\beta \omega_1)^3} = 0 \]  
(41)

\[ R_2 = \frac{\beta \epsilon_j^2 E_o (1-\beta \omega_2)(1-\beta \omega_3)}{(1-\beta \omega_1)^2} \]  
(42)

\[ R_3 = \frac{\beta \epsilon_j^2 E_o (1-\beta \omega_2)^2 (1-\beta \omega_3)}{(1-\beta \omega_1)^2} \]  
(43)

From symmetry, \( \omega_2 = \omega_3 = \omega \) and \( \omega_1 = 0 \) by virtue of Equation (41), yields

\[ R_2 = R_3 = \beta \epsilon_j^2 E_o (1-\beta \omega)^3 \]  
(44)

and

\[ (R_i R_i)^{1/2} = \sqrt{2} \beta \epsilon_j^2 E_o (1-\beta \omega)^3 \]  
(45)

and

\[ \begin{bmatrix} \frac{\partial f}{\partial R_i} \end{bmatrix} = 0 \begin{bmatrix} 1 \\ \sqrt{2} \\ \sqrt{2} \end{bmatrix} \]  
(46)

Differentiating \( R_i \) with respect to \( \omega_i \) and \( \epsilon_i \) and substituting along with (46) into (31) yields

\[ d\lambda = \frac{2\sqrt{2} \beta \epsilon_j E_o (1-\beta \omega)^3 d\epsilon_j}{H + 3\beta^2 \epsilon_j^2 E_o (1-\beta \omega)^2} \]  
(47)

Finally, Equation (33) yields incremental stress-strain relationship for strain control testing as

\[ d\sigma = \left( E_o (BW)^4 - \frac{8\beta^2 \epsilon_j^2 E_o^2 (BW)^6}{H + 3\beta^2 \epsilon_j^2 E_o (BW)^2} \right) d\epsilon_j \]  
(48)

where \( BW = (1-\beta \omega) \).
Similarly, it can be shown that proceeding in a similar way incremental stress-strain relationship for stress control testing can be expressed as

\[
d_1 = \begin{cases} 
\frac{1}{E_o (BW)^4} + \frac{8\beta^2 \sigma^2/(BW)^{10} + E_o^2}{H - \frac{5\sigma^2}{E_o (BW)^6}} d\sigma 
\end{cases} \tag{49}
\]

and for strain control testing

\[
d_1 = \begin{cases} 
\frac{1}{E_o (BW)^4} + \frac{4\alpha^2 \sigma^2/E_o^2 (AW)^6}{H - \frac{3\sigma^2}{E_o (AW)^4}} d\sigma \end{cases} \tag{50}
\]

4.2 Uniaxial tension

Following the same logic as in the case of uniaxial compression, it can be shown that for strain control testing

\[
d\sigma = \left( E_o (AW)^2 - \frac{4(\alpha^2 \varepsilon_i^2 E_o^2 (AW)^2)}{H + \alpha^2 \varepsilon_i^2 E_o} \right) d_1 \tag{51}
\]

where \( AW = (1-\alpha_1) \), and for stress control testing

\[
d_1 = \begin{cases} 
\frac{1}{E_o (AW)^4} + \frac{4\alpha^2 \sigma^2/E_o^2 (AW)^6}{H - \frac{3\sigma^2}{E_o (AW)^4}} d\sigma \end{cases} \tag{51}
\]

4.3 Biaxial compression

Predictive ability of the proposed elasto-damage model for a multi-axial stress path is investigated for the biaxial stress state defined by \( \sigma_1 = \sigma_2 = \sigma \). The 2x2 compliance and stiffness matrices for a general two-dimensional state of stress as a special case of Equations (1) are given by

\[
\bar{C}_{ij} = \bar{C}_{ji} \hspace{1cm} (i, j = 1, 2) \text{ of Equation (1)}
\]

whose inverse may be written as:

\[
\bar{D}_{11} = E_o \frac{(1-\alpha_1)^2 (1-\beta_0)^2 (1-\gamma_0^1)^2}{(1-\gamma_0^1)^2} \times \frac{1}{(1-\gamma_0^1)^2 (1-\gamma_0^2)^2 - \nu^2} \tag{52}
\]

\[
\bar{D}_{22} = E_o \frac{(1-\alpha_2)^2 (1-\beta_0)^2 (1-\gamma_0^2)^2}{(1-\gamma_0^2)^2} \times \frac{1}{(1-\gamma_0^1)^2 (1-\gamma_0^2)^2 - \nu^2}
\]

\[
\bar{D}_{12} = \bar{D}_{21} = E_o \nu (1-\alpha_1) (1-\alpha_2) (1-\beta_0) \]

Proceeding in a manner analogous to the uni-dimensional stress state, following is the resulting sequence of salient equations:

\[
\rho \Lambda = \frac{\sigma^2}{E_o (1-\beta_0)^2} \left( 1 - \frac{\nu}{(1-\gamma_0^1)(1-\gamma_0^2)} \right) \tag{53}
\]

\[
\varepsilon_1 = \frac{\sigma_1}{E_o (1-\beta_0)^2} \left( 1 - \frac{\nu}{(1-\gamma_0^1)(1-\gamma_0^2)} \right) \tag{54}
\]

Using \( \sigma_1 = \sigma_2 = \sigma \), the inversion of Equation (54) yields

\[
\sigma = \frac{\varepsilon_1 E_o (1-\beta_0)^2}{\nu (1-\gamma_0^1)(1-\gamma_0^2)} \tag{55}
\]

Also

\[
R_1 = R_2 = -\frac{2\nu \sigma^2}{E_o (1-\gamma_0^1)^2 (1-\gamma_0^2)} = 0 \tag{56}
\]

Components \( R_1 \) and \( R_2 \) = 0 due to the non-negativity of \( \alpha \).

Also

\[
R_3 = \frac{2\beta \varepsilon_1^2 E_o (1-\beta_0)}{(1-\nu)} \tag{57}
\]

Following the same steps as for uniaxial case, together with \( \omega_1 = \omega_2 = 0 \) and \( \omega_3 = \omega \), leads to the following incremental \( \sigma-\varepsilon \) relationship for equal biaxial compression

\[
d\sigma = \begin{cases} 
\frac{E_o (1-\beta_0)}{1-\nu} - \frac{8\beta^2 \varepsilon_1^2 E_o^2 (1-\beta_0)^2 (1-\nu)^2}{H + \frac{2\beta^2 \varepsilon_1^2 E_o}{(1-\nu)}} \end{cases} d\varepsilon \tag{58}
\]

This form is useful for strain control, however, for stress control the procedure will be similar to that for uniaxial case. The incremental stress-strain relationship for equal biaxial compression is expressed as
5 Determination of Regression Coefficients

The parameters $\alpha$, $\beta$ and $\gamma$ are functions of initial modulus of elasticity, $E_o$, uniaxial compressive strength, $f'_c$, and normalized strain invariants, $I_1$, $J_2$ and $e_3$ (where $I_1 = \varepsilon_{kk} & e_3$ and $J_2 = \frac{1}{2} tr (e_{ij})$). Here, $\varepsilon_3$ and $e_3$ represent the minor principal and deviatoric strain, respectively. The suggested forms of $\alpha$, $\beta$ and $\gamma$ are as follows:

$$\alpha = \alpha_0 (f'_c, E_o) + \alpha_1 (f'_c, E_o) \frac{I_1}{\varepsilon_3} + \alpha_2 (f'_c, E_o) \frac{J_2}{e_3}$$

or

$$\alpha = 0 \quad (\sigma_1, \sigma_2 < 0)$$

$$\alpha = \alpha (f'_c, E_o) \quad (\sigma_1, \sigma_2 \geq 0) \quad \text{(path independent)} \quad (60)$$

Phenomenological evidence is the essential guide in deciding if a particular stress state corresponds to a path dependent or path independent loading $I_1$, $J_2$ computed at the beginning of load increment.

$$\beta = \beta_0 (f'_c, E_o) + \beta_1 (f'_c, E_o) \frac{I_1}{\varepsilon_3} + \beta_2 (f'_c, E_o) \frac{J_2}{e_3}$$

or

$$\beta = 0 \quad (\sigma_1, \sigma_2 > 0)$$

$$\beta = \beta (f'_c, E_o) \quad (\sigma_1 > 0, \sigma_2 < 0) \quad \text{(path dependent)} \quad (61)$$

and

$$\gamma = \gamma (f'_c, E_o) \quad (\sigma_1, \sigma_2 < 0) \quad \text{(path independent)} \quad (62)$$

where,

$$\alpha_i = \alpha_i + \alpha_i f'_c + \alpha_i f'_c^2 + \alpha_i E_o, \quad (i = 0, 1, 2, 3)$$

path dependent \hspace{1cm} (63)

$$\beta_i = \beta_i + \beta_i f'_c + \beta_i f'_c^2 + \beta_i E_o, \quad (i = 0, 1, 2, 3)$$

path independent \hspace{1cm} (64)

These parameters were calibrated by regressing against experimental $\sigma$–$\varepsilon$ data for different $f'_c$. The range of $f'_c$ used in regression varies from 4,000 psi to 17,400 psi, and is listed in Table 1. Stress paths used in regression are listed in Table 2.

<table>
<thead>
<tr>
<th>$f'_c$ (psi)</th>
<th>$E_o (10^6$ psi)</th>
<th>$\nu_o$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4000</td>
<td>3.605</td>
<td>0.19</td>
</tr>
<tr>
<td>9434</td>
<td>6.023</td>
<td>0.19</td>
</tr>
<tr>
<td>13062</td>
<td>6.444</td>
<td>0.19</td>
</tr>
<tr>
<td>17416</td>
<td>7.126</td>
<td>0.19</td>
</tr>
</tbody>
</table>

ACI Equation $E_c = 57000 \sqrt{f'_c}$ ($f'_c$ in psi) $= 4730 \sqrt{f'_c}$ ($f'_c$ in MPa)

Wee et al. [32] $E_c = 102000 (f'_c)^{1/3}$ ($f'_c$ in MPa)

Table 1: Range of $f'_c$ used in regression

<table>
<thead>
<tr>
<th>$\sigma_1/\sigma_2$ for $\alpha$</th>
<th>$\sigma_1/\sigma_2$ for $\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1$</td>
<td>$0$</td>
</tr>
<tr>
<td>$-0.25$</td>
<td>$0.2$</td>
</tr>
<tr>
<td>$-0.125$</td>
<td>$0.5$</td>
</tr>
<tr>
<td>$-0.1$</td>
<td>$0.8$</td>
</tr>
<tr>
<td>$-0.075$</td>
<td>$1.0$</td>
</tr>
<tr>
<td>$-0.05$</td>
<td></td>
</tr>
<tr>
<td>$-0.025$</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Stress paths for determination of $\alpha$, $\beta$

Regression was carried out in two steps as illustrated by Desai [30] and shown in Figure 1. In the first step, for a particular $f'_c$ and $E_o$, $\alpha$ or $\beta$ is determined by matching the peak stress to experimental data for varying stress ratios $\sigma_i/\sigma_2$. Then this data is regressed using the proposed form of $\alpha$ or $\beta$ (Equations (60) & (62)) to obtain $\alpha_0$, $\alpha_1$, $\alpha_2$, $\alpha_3$ or $\beta_0$, $\beta_1$, $\beta_2$, $\beta_3$. At the end of this step, one has a set of $\alpha_i$’s and $\beta_i$’s, where $i = 0, 1, 2, 3$. 

$$\gamma = 0 \quad \text{for all other stress states}$$
This process is then repeated for all the listed $f_c'$. The sets of $\alpha_i$’s and $\beta_i$’s are then further regressed using the proposed functional forms of $\alpha_i$’s and $\beta_i$’s (Equation (65)) to obtain $\alpha_i$, $\beta_i$, $\gamma_i$, $\alpha_i$, $\beta_i$, $\gamma_i$, $\alpha_i$, $\beta_i$, $\gamma_i$, $\alpha_i$, $\beta_i$, $\gamma_i$.

For determination of $\gamma_i$’s and the path independent $\bar{\alpha}, \bar{\beta}$, regression is carried only once, since these parameters are presumed to be a function only of $f_c'$ and $E_o$.

Final form of $\alpha_i$’s, $\beta_i$’s and $\gamma$ is as follows:

- **$\alpha$ for path-dependent stress states**
  \[
  \alpha_0 = -1.989E+01 + 1.788E-03 * f_c' - 5.644E - 08 * f_c'^2 - 4.758E - 07 * E_o
  \]
  \[
  \alpha_1 = -3.495E + 01 - 1.445E - 04 * f_c' - 4.415E - 09 * f_c'^2 + 4.593E - 06 * E_o
  \]
  \[
  \alpha_2 = 2.825E + 01 - 2.561E - 03 * f_c' + 8.123E - 08 * f_c'^2 - 6.864E - 07 * E_o
  \] (67)

- **$\beta$ for path-dependent stress states**
  \[
  \beta_0 = 3.647E + 01 - 2.641E - 05 * f_c' + 8.2457E - 10 * f_c'^2 - 1.6974E - 08 * E_o
  \]
  \[
  \beta_1 = -6.7164E - 02 + 6.008E - 06 * f_c' - 1.712E - 10 * f_c'^2 + 8.3786E - 10 * E_o
  \]
  \[
  \beta_2 = -1.2177E - 01 - 1.5589E + 05 * f_c' - 4.1993E - 10 * f_c'^2 - 5.7088E - 09 * E_o
  \]
  \[
  \beta_3 = 6.1923E - 02 + 8.7543E - 06 * f_c' + 2.3097E - 10 * f_c'^2 + 4.3012E - 09 * E_o
  \] (69)

- **$\beta$ for path-independent stress states**
  \[
  \beta = 2.516E - 01 - 1.978E - 05 * f_c' + 6.421E - 10 * f_c'^2 - 1.152E - 08 * E_o
  \] (70)

- **$\gamma$ for path-independent stress states**
  \[
  \gamma = 1.5863 - 1.221E - 04 * f_c' + 3.4586E - 09 * f_c'^2 + 5.3091E - 08 * E_o
  \] (71)

Depending on the state of stress $(\sigma_1, \sigma_2)$, the following combinations of $\alpha, \beta$ and $\gamma$ are proposed as shown in Table 3.

### 6 Results

Results predicted by the proposed model discussed in earlier sections are verified for uniaxial and biaxial loading conditions by comparing them with the available experimental results. A strain control programme is used for plotting curves for predicted stress-strain response, volumetric dilatation and apparent Poisson’s ratio for
State of stress | $\alpha$ | $\beta$ | $\gamma$
--- | --- | --- | ---
$\sigma_1 > 0, \sigma_2 \geq 0$ | $\overline{\alpha}(f'_c, E_o)$ | 0 | 0
$\sigma_1 > 0, \sigma_2 < 0$ | $\alpha \left( f'_c, E_o, \frac{I_1}{e_3}, \frac{J_2}{e_3} \right)$ | $\overline{\beta}(f'_c, E_o)$ | 0
$\sigma_1 \leq 0, \sigma_2 \leq 0$ | 0 | $\beta \left( f'_c, E_o, \frac{I_1}{e_3}, \frac{J_2}{e_3} \right)$ | $\gamma(f'_c, E_o)$

Table 3: Combinations of $\alpha$, $\beta$ and $\gamma$

Three different concretes (concrete A, $f'_c = 4,000$ psi (27.6 MPa), concrete B, $f'_c = 9,434$ psi (65 MPa) and concrete C, $f'_c = 17,416$ psi (120 MPa) were selected for comparison of predicted results of the model under uniaxial and biaxial loading conditions. For the case of uniaxial compression, comparison with experimental results of Wischers [31] and Wee et al. [32] shows that the predicted curves for concretes A and B (Figures 3 and 4) are stiffer and less ductile than the experimental curves i.e. the predicted peak stress compares quite well but the strains are somewhat on the lesser side. This can be attributed to the absence of plastic strains. Additional plastic strains will shift the curves to the right, making it comparable with the experimental values. Curve for concrete C, Figure 5, is in close agreement with the experimental curve [32] both in terms of peak stress and peak strain. It is evident from Figures 3–5 that the model can capture the increasing brittleness and decreasing ductility with the increase in uniaxial compressive strength quite effectively.

Figures 3–7 reflect the predictive capability of the model to capture the phenomenological behaviour of volumetric dilatation. In Figures 3–5, curves for lateral strains are shown in which the increase in lateral strains as the stress reaches the peak stress is clearly visible. This increase in lateral strains is due to the dramatic increase in apparent Poisson’s ratio (Figure 6), which results in the change of sign of dilatancy (volumetric strain) as shown in Figure 7. It can be seen from the figure that under compression concrete always exhibits this phenomenon irrespective of the stress paths. Figure 6 represents the change in Poisson’s ratio for all the three concretes. The curves are similar to the ones predicted by Krajcinovic and Fonseka [15] and Voyiadjis and Abu-Lebdeh [25]. Curves predicted in the present study are more consistent with the observed behaviour since the change in Poisson’s ratio is more evident near the peak stress, while the change in Poisson’s ratio is evident right from the onset of damage in the curves predicted by Krajcinovic and Fonseka [15] and Voyiadjis and Abu-Lebdeh [25].

For the case of uniaxial tension, softening in post-peak zone of stress-strain plots for all the concretes has been captured by the model effectively as shown in Figure 8. Shapes of the stress-strain plots correlate with the experimental results of Gopalaratnam and Shah [33].

For the case of biaxial compression, comparison with the experimental results of Kupfer et al. [34], Figures 9 and 10, indicates that the predicted peak stress envelope is in close agreement with the experimental peak stress.

It should be noted that this remarkable correlation with experimental data was obtained by introducing three parameters ($\alpha$, $\beta$ and $\gamma$) in the effective compliance matrix $\tilde{C}$, on the basis of available 2-D data for concrete. Since it is already shown that this rather simple approach is capable of capturing almost all the observed phenomenological trends of concrete, incorporation of more three dimensional data will yield a more rigorous model capable of predicting multiaxial behaviour of concrete.

7 Conclusions

An elasto-damage bounding surface model for monotonic behaviour of normal and high strength concrete is developed in this paper. A generalized compliance matrix is proposed in the principal coordinate system based on phenomenological evidences and physical insight. Typical behavioural trends of concrete are taken into account by
introducing three new parameters $\alpha$, $\beta$ and $\gamma$ in the proposed compliance matrix.

Results presented in the paper demonstrate that the proposed model predicts the behaviour of concrete under multiaxial monotonic loadings adequately. Almost all the essential features of concrete including the volumetric dilatation are captured by the model. Absence of plastic strains has resulted in stiffer stress-strain curves, therefore, it is recommended to include the effect of plastic deformation in modelling the behaviour of concrete.

In order to make the model general and applicable to real three dimensional problems, work is in progress and more triaxial data will be incorporated in determining the regression coefficients.
Figure 5: Stress-strain curves for uniaxial compression
\( f'_c = 17416 \text{ psi} \)

Figure 6: Apparent Poisson’s ratio

Figure 7: Volumetric dilatation for different stress paths
\( f'_c = 4000 \text{ psi} \)

Figure 8: Stress-strain curves for uniaxial tension

\( f'_c = 9434 \text{ psi} \)

\( f'_c = 17416 \text{ psi} \)
Acknowledgement

The authors are indebted to the Department of Civil Engineering, King Fahd University of Petroleum & Minerals, for their support in the pursuit of this work.

References
