Nominal Moment Strength $M_n$ of Rectangular Sections
Having Both Tension and Compression Reinforcement

Doubly reinforced Sections: are those sections of concrete in which both tension and compression steel are used.

Singly reinforced Sections: The section contains only tension reinforcement.

Why use doubly reinforced sections?
- To reduce size of the cross-section
- To reduce deflection
- When it is hard to place tension steel in width
- To control deflection against shrinkage and creep.

Doubly Reinforced Section

\[ P = \frac{A_s}{bd} \quad \text{(tension steel reinforcement ratio)} \]
\[ P' = \frac{A_{s'}}{bd} \quad \text{(compression steel reinforcement ratio)} \]

The stress in compression steel is calculated based on the strain at 0.5 level, which is compatible with strain in concrete: $\varepsilon_c = 0.003$ (using similar triangle).
Remark 1:
There are two types of problems, one is called investigation problem and the other is called design problem. The first is concerned with finding strength capacity given cross section and steel amount, while the other focuses on selection of beam size and steel amount.

Remark 2:
The computation of doubly reinforced sections is the same as that for singly reinforced except that the compressive force \( C \) consists of two parts: one in concrete and the other is on compression steel.

\[
T = A_s f_y
\]

\[
C = 0.85 f'_c b a - 0.85 f'_c A'_s + f_s' A'_s
\]
acting at \( \frac{a}{2} \)
acting at \( d' \) from top

\[
C = C_c + C_s
\]

\[
C_c = 0.85 f'_c a b
\]

\[
C_s = A'_s (f'_s - 0.85 f'_c)
\]

\[
M = C_c [d - \frac{a}{2}] + C_s [d - d']
\]
Maximum tension steel permitted under AEC - 10.3.3.

* For tension steel \( p \leq p_{\max} = 0.75 \, p_b \)

* For compression steel \( p \leq p_b \)

Notice that both tension and compression steel will be granted its yield with the above conditions.

Ex. 2.10.1: For the section given below, determine \( M_0 \) and check if the amount of steel \( A_s \) complies with AEC - 10.3.3.

\[ A_s = 2 \# 8 \]

\[ A_s = 8 \# 10 \]

\[ A_s = 2.4 \] \( \text{in}^2 \)

\[ A_s = 1.58 \] \( \text{in}^2 \)

Data:

\[ f' = 3000 \, \text{psi} \]

\[ f_y = 60,000 \, \text{psi} \]

\[ b = 14'' \]

\[ d = 26'' \]

\[ d' = 3'' \]

\[ A_s = 10.16 \, \text{in}^2 \]

\[ A_s' = 1.58 \, \text{in}^2 \]

Assume that both tension steel and compression steel have yielded \((f_s = f_y \text{ and } f_s' = f_y)\).
\[ T = A_{uy} f_y = (10.16)^2 (60 \frac{\text{kips}}{\text{in}^2}) = 610 \text{ kips} \]

\[ C_c = 0.85 f'_{cub} A = (0.85)(5) \frac{\text{kips}}{\text{in}^2} (14) \text{ in} \cdot A = 59.5A \]

\[ C_s = A_s (f'_{fs} - 0.85 f'_{cub}) = 1.58 (60 - 0.85(5)) = 88 \text{ kips} \]

\[ T = C = C_c + C_s \\
610 = 59.5A + 88 \Rightarrow \quad A = 8.17 \text{ in} \]

\[ x = \frac{a}{\beta} = \frac{8.77}{18} = 0.49'' \]

Check if tension and compression steel have yielded. By comparing \( \epsilon_s \) and \( \epsilon_s' \) with
\[ \epsilon_s = \frac{f_y}{E} = \frac{60}{29000} = 0.00207 \]

1. \[ \epsilon_s = \frac{0.003}{10.96} \Rightarrow \quad \epsilon_s = 0.004 > 0.00207 \]
   So tension steel has yielded.

2. \[ \epsilon_s' = \frac{0.003}{7.96} \Rightarrow \quad \epsilon_s' = 0.00218 > 0.00207 \]
   So compression steel has yielded.

Our assumption is correct and the forces are
\[ T = 610 \text{ kips} \]
\[ C_c = 522 \text{ kips} \quad (\approx 59.5 \times 8.77) \]
\[ C_s = 88 \text{ kips} \]
\[ M_n = C_c \left( d - \frac{a}{2} \right) + C_s (d - d') \]
\[ = 522 \left( 26 - \frac{8.77}{2} \right) \left( \frac{1}{12} \right) + 88 \left( 26 - 3 \right) \left( \frac{1}{12} \right) \]
\[ = 940 + 169 \]
\[ M_n = 1109 \text{ ft-kips} \]

It can be shown that \( x_b = 15.39 \)

\[ \begin{align*}
I_b & \quad \frac{x_b}{26-x_b} = \frac{0.003}{0.00207} \\
& \quad 0.00207
\end{align*} \]

\[ x_b = 15.39 \]
\[ 0.75x_b = 11.54 \]
\[ x = 10.96 \]

\[ \therefore x < 0.75x_b \quad (\because \text{It complies with ACI-10.3.3}) \]
Ex. 3.10.2  Repeat the solution of previous problem except that $A_s = \sqrt{4} \# 11$ instead of $A_s = \# 11$.

$A_s = 2 \# 8$

Data: $f_c' = 3000 \text{ psi}$, $f_y = 60,000 \text{ psi}$,
$b = 14''$, $d = 26''$, $d' = 3''$

$A_s' = 1.58$, $A_s = 6.24$

Assume both tension and compression steel have yielded

$T = A_s f_y = 60 (6.24) = 374 \text{ kips}$

$C_c = 0.85 f_c' a b = 59.5 \text{ kips}$

$C_s = A_s' (f_c' - 0.85 f_c') = 88 \text{ kips}$

$T = C_c + C_s \Rightarrow \varepsilon_a = 4.81 \text{ in}$

$X = \frac{a}{p} = 6.01$

$\frac{\varepsilon_s}{0.003} = \frac{19.99}{6.01} \Rightarrow \varepsilon_s = 0.009 > \varepsilon_y$

- Tension steel has yielded

$\frac{\varepsilon_s'}{3.01} = \frac{0.003}{6.01} \Rightarrow \varepsilon_s' = 0.0015 < \varepsilon_y'$

- Compression steel has not yielded

- $C_c$ and $C_s$ were not right.

and also $\varepsilon_s'$ is not correct because it has been calculated using wrong $A_s$.

Remark: for doubly reinforced section, you use $A_s'$ after you use the limit of $A_s$ (little $A_s$ will not allow $A_s'$ to yield).
Determine the location of N-A (i.e. X = ?)

\[ T = A_{f y} = 60 \times (6.24) = 374.4 \]

\[ C_e = 0.85 f_y b \left( \frac{8}{X} \right) = 47.6 X \]

\[ C_s = A'_s \left( f'_s - 0.85 f_y \right) = 1.58 \left( \frac{29000}{X} \right) \left( \frac{X-3}{0.003} \right) = 4.25 \]

\[ T = C_e + C_s \]

\[ 374.4 = 47.6 X + 137.46 \left( \frac{X-3}{X} \right) = 6.715 \]

\[ 374.4 X = 47.6 X^2 + 137.46 - 412.38 - 6.715 X \]

\[ 47.6 X^2 - 248.52 X - 412.4 = 0 \]

\[ X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

\[ X = 6.46 \]

\[ a = 5.17 \]

\[ C_e = 47.6 (6.46) = 307.5 \]

\[ C_s = 66.9 \text{ (Substituting X into C_s)} \]

\[ M_n = 307.6 \left[ \frac{26 - 0.5 (5.17)}{12} \right] + 66.9 \left( \frac{26 - 3}{18} \right) = 728 \text{ ft-lbs} \]

\[ \text{Little As} \]

\[ \text{max As} \]
For example. 3.10.2, determine $M_n$ if compression steel had not exist (i.e. $A_{s'} = 0$)

\[ a = \frac{A_{sfy}}{0.85}\frac{b}{6} = 6.29'' \]

\[ M_n = A_{sfy} \left[ d - \frac{a}{2} \right] \]

\[ M_n = (6.24)(60) \left[ 26 - \frac{6.29}{2} \right] \left( \frac{1}{12} \right) = 713 \text{ ft-kips} \]

- When both $A_s$ and $A_{s'}$ are present, \( M_n = 728 \text{ ft-kips} \)
- When only $A_s$ is present, \( M_n = 713 \text{ ft-kips} \)

The extra moment percentage that is carried by $A_{s'}$ is:

\[ \frac{728 - 713}{713} \times 100 = 2.1\% \]

Remark:

As a conclusion, one can state that $A_{s'}$ only carries a small portion of $M_n$ and most of it is carried by $A_s$. This is why in doubly reinforced concrete design, one has to design as singly reinforced section with the max $P$ and if there is still moment to be carried then provide $A_{s'}$. 
Ex. 3.11.2. Determine \( A_s \) and \( A_s' \) required to carry a service live load moment of 390 ft-kips and a service dead load moment of 200 ft-kips. Using \( b = 18\text{ in} \), \( d = 26\text{ in} \), \( d' = 3\text{ in} \), \( f_c' = 5000 \text{ psi} \), \( f_y = 60,000 \text{ psi} \), use the maximum allowed \( A_s \) as per ACI-318-05.

\[
\max P \text{ is when } x = 0.75 x_b
\]

Balanced condition:

\[
\frac{x_b}{d} = \frac{0.003}{0.003 + \frac{60}{29}} \Rightarrow x_b = 15.39 \text{ in}
\]

\[
x = 0.75 x_b = 11.54
\]

\[
a = 9.232
\]

\[
C = 0.85 \times 5 \times 9.232 \times 14 = 549 \text{ kips}
\]

\[
T = A_s f_y = 60 A_s
\]

\[
C \cdot \frac{C}{A_s} = \frac{549}{60} = 9.15 \text{ in}^2
\]

\[
M_n = 549 \left[ 26 - \frac{1}{2} (9.232 \times 14) \right] = 978 \text{ ft-kips}
\]

\[
M_n = 1048 \text{ ft-kips}
\]
So far we are dealing with \( T_C \) and \( C_C \). The extra moment is going to be carried by \((T_S+T_C)\) couple.

\[
M_n = M_n^C + M_n^S
\]

\[1048 = 978 + M_n^S \Rightarrow M_n^S = 70 \text{ ft-kips}
\]

\[
M_n^S = C_S(d - d_f)
\]

\[70(12) = C_S(26-3) \Rightarrow C_S = 36.5 \text{ kips}
\]

\[
C_S = A_S'(f_s' - 0.85f_c')
\]

Assume compression steel has yielded:

\[36.5 = A_S'(60 - 0.85f_c') \]

\[\Rightarrow A_S' = 0.65 \text{ in}^2\]

How do you check if comp. steel has yielded?

\[
A_s = \frac{T}{f_y} = \frac{T_S + T_C}{f_y} = \frac{36.5 + 54.9}{60} = 0.976 \text{ in}^2
\]

\[
A_s = 9.76 \text{ in}^2 \quad \text{&} \quad A_S' = 0.65 \text{ in}^2
\]