Chapter 6.

Shear Stress
Shearing Stresses in Beams

Laminated beam

Unbonded lamina

Glued lamina

\[ \tau_{yx} = \text{horizontal shearing stress} \]

\[ \tau_{xy} = \text{transverse shearing stress (vertical shearing stress at the cross section)} \]

\[ \tau_y = \tau_x \]
Shear Stresses in Beams

Note:

Shear stress \( \tau \) at a given cross section is due to shear force \( V \), which will not exist unless moment is changing (i.e. \( \frac{dM}{dx} \neq 0 \)).

\[
R(x) = \int \sigma(x) \, dA = \int \frac{-M(x)}{I} \, y \, dA = \frac{-M(x)}{I} \int y \, dA
\]

\[
R(x + \Delta x) = \int \sigma(x + \Delta x) \, dA = \int \frac{-M(x + \Delta x)}{I} \, y \, dA = \frac{-M(x + \Delta x)}{I} \int y \, dA
\]

Let \( Q = \int y \, dA \) = first moment of inertia \( I * \bar{y} \)

\[
R(x) = \frac{-M(x)}{I} Q
\]

\[
R(x + \Delta x) = \frac{-M(x + \Delta x)}{I} Q
\]

\[
\frac{dR}{dx} = \lim_{\Delta x \to 0} \frac{R(x + \Delta x) - R(x)}{\Delta x} = \frac{-Q}{I} \lim_{\Delta x \to 0} \frac{M(x + \Delta x) - M(x)}{\Delta x}
\]

\[
\frac{dR}{dx} = -\frac{VQ}{I} \quad \text{(A)}
\]
\[ q = \text{force/unit length and constant across width (shear flow)} \]

\[ \Sigma F_x = 0 \quad -R(x) + q \Delta x + R(x+\Delta x) = 0 \]

\[ \frac{R(x+\Delta x) - R(x)}{\Delta x} = -q \]

Taking limit of both sides:

\[ \frac{dR}{dx} = -q \quad \text{①} \]

\[ \text{①} = \text{②} \quad \Rightarrow \quad q = \frac{V\phi}{I} \]

\[ \Rightarrow \quad \tau = \frac{q}{t} = \frac{q}{b} \]

\[ \tau = \frac{V\phi}{Ib} \]
The beam is subjected to shear force $V = 15$ kN. Find $T_A$ and $T_B$ and show these stresses over elements $A$ and $B$.

\[ Q = A \cdot \bar{y} \]

1. Locate centroid and draw $y-z$ axis through centroid.

\[ \bar{y} = \frac{\sum A_i \cdot y_i}{\sum A_i} = \frac{(250)(30) + 250(20)(15) + 125(20)}{(250)(30) + 250(20) + 125(30)} = 174.7 \text{ mm} \]

2. Calculate moment of inertia about $z$-axis (or N.A.)

\[ I_z = 0.21818 \times 10^{-3} \]

\[ \Omega = \frac{V \phi}{I_b} \]

\[ T_A = \frac{\sqrt{\Omega_A}}{I_{b_A}} = \frac{(15,000)(0.2 \times 0.03 \times [0.1353 - 0.015])}{(0.21818 \times 10^{-3})(0.025)} = 1.99 \text{ MPa} \]

\[ T_B = \frac{\sqrt{\Omega_B}}{I_{b_B}} = \frac{(15,000)(0.125 \times 0.03 \times [0.1747 - 0.015])}{(0.21818 \times 10^{-3})(0.025)} = 1.65 \text{ MPa} \]

Note: arrows meet by their heads or tail.
Plot variation of $Q$ and $z$ as a function of depth.

$$Q = \frac{\pi}{2} \left[ y^2 - \frac{b}{2} \right]$$

Possible $z_{\text{max}}$ location
Remark

Always think about the internal vertical shear force \( V_y \) as a source for creating shear stress (indirectly through the change of moment) in two \( \perp \) planes; one is the vertical plane where \( V_y \) acts and the other is the horizontal plane to keep \( \Sigma F_x = 0 \). These two shear stresses are equal at their intersection as shown below.

![Diagram showing vertical and horizontal planes]

Note that for pure moment (\( M = \) constant), \( V = 0 \) accordingly the shear stresses in vertical and horizontal planes are zero and therefore no slipping and no need for nails or glue.

Objective

1. How to find the value of \( p \) or \( \tau \) at any horizontal level.
2. How to find shear stress at glue and force carried by nail.
3. How to find the maximum shear stress due to \( V \) at a given cross section.

\[
F_{\text{carried by nail}} = 9.5
\]

\[
p = \frac{V \cdot q}{I} = \left( \frac{V}{I} \right) q
\]

\[
\tau = \left( \frac{V}{I} \right) \left( \frac{q}{b} \right)
\]

\[
\tau_{\text{max}} = \left( \frac{V}{I} \right) \left( \frac{q}{b} \right)_{\text{max}}
\]
For the beam shown below, find the following:

1. Glue strength (shear stress at glue level)
2. The required nail strength at spacing of 50mm
3. The maximum shear stress
4. Shear stress distribution

---

Finding Centroid and moment of inertia

\[
\overline{Z} = 80 \text{ mm}
\]

\[
\overline{Y} = \frac{\sum A_i y_i}{\sum A_i}
\]

\[
\overline{Y} = \frac{(160 \times 20 \times 13) + 160 \times 20 \times 7 + 20 \times 5 \times 10}{160 \times 20 + 160 \times 20 + 20 \times 5}
\]

\[
\overline{Y} = 91.3 \text{ mm}
\]

<table>
<thead>
<tr>
<th>Shape</th>
<th>( I_{z_i} )</th>
<th>( A_i )</th>
<th>( d_i )</th>
<th>( I_{z_i} + A_i d_i^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \frac{1}{12} (20)^3 ) (160)</td>
<td>160 \times 20</td>
<td>38.7</td>
<td>4.9 \times 10^6</td>
</tr>
<tr>
<td></td>
<td>( \frac{1}{12} (160)^3 ) (20)</td>
<td>160 \times 20</td>
<td>71.3</td>
<td>2.57 \times 10^6</td>
</tr>
<tr>
<td></td>
<td>( \frac{1}{12} (20)^3 \times 50 )</td>
<td>50 \times 20</td>
<td>81.3</td>
<td>6.64 \times 10^6</td>
</tr>
</tbody>
</table>

\[
I_Z = 14.1 \times 10^6 \text{ mm}^4 = \text{the sum of last column}
\]

\[
I_Z = 14.1 \times 10^{-6} \text{ m}^4
\]
6.9 Stresses in Thin-Walled Pressure Vessels

* Analysis is limited to cylindrical and spherical vessels.

\[ \Sigma F_x = 0 : \quad 2\pi rt \sigma_2 - \rho \pi r^2 = 0 \]
\[ \Rightarrow \sigma_2 = \frac{pr}{2t} \]

\[ \Sigma F_y = 0 : \quad 2\pi \sigma_2 t + 4\pi r \sigma_1 \Delta x t = 0 \]
\[ \Rightarrow \sigma_1 = \frac{pr}{2t} \]

Summary: For cylindrical vessel subjected to internal pressure \( P \) of radius \( r \) and thickness \( t \), the following stresses will be developed:

\[ \sigma_1 = \text{hoop stress} = \frac{pr}{t} \]
\[ \sigma_2 = \text{longitudinal stress} = \frac{pr}{2t} \]
\[ \sigma_3 = \text{maximum stress} = \frac{pr}{2t} \]
Summary: For a sphere with radius \( r \) and thickness \( t \) the stresses which will develop due to internal pressure \( P \) are:

\[
\sigma_1 = \sigma_2 = \frac{Pr}{2t}
\]

\[
\sigma \leq \frac{Pr}{4t}
\]

Data: \( r = 250 \text{ mm}, \quad t = 6 \text{ mm}, \quad \sigma_{ult} = 400 \text{ MPa}, \quad P = 5.5 \text{ MPa}

Find \( F.S. \):

\[
\sigma_1 = \frac{P}{t} = \frac{(5.5) (250)}{6} = 114.58
\]

\[
F.S. = \frac{\sigma_{ult}}{\sigma_{cal.}} = \frac{400}{114.58} = 3.49
\]

For a cylinder with weld:

\( r = 295 \text{ mm}, \quad t = 5 \text{ mm}, \quad P = 4 \text{ MPa}

Find \( \sigma \) and \( \tau \) at weld:

\[
\sigma_1 = \frac{P}{t} = 236 \text{ MPa}
\]

\[
\sigma_2 = \frac{P}{2t} = 118 \text{ MPa}
\]

\( \tau_{da} = (\sigma_1 \cos 65^\circ \text{ da}) \cos 65^\circ + (\sigma_2 \cos 25^\circ \text{ da}) \cos 25^\circ \Rightarrow \tau = 139 \text{ MPa} \)

\( \tau_{da} = (\sigma_1 \cos 65^\circ \text{ da}) \sin 65^\circ + (\sigma_2 \cos 25^\circ \text{ da}) \sin 25^\circ \Rightarrow \tau = 45.2 \text{ MPa} \)
**Compound Stresses**

**Compound normal stress:** Normal stress caused by different forces and moments such as $(N, M_y, M_z)$

**Compound shear stress:** Shear stresses caused by different forces and moments such as $(V_y, V_z, T)$

See Figure below.

\[
\sigma = \sigma(N, M_y, M_z)
\]

\[
\tau = \tau(V_y, V_z, T)
\]

\[
\sigma = \pm \frac{N}{A} \pm \frac{M_z y}{I_z} \pm \frac{M_y z}{I_y}
\]

Select sign whenever moment is causing tension in the $y$ or $z$-axis.
Problem

(a) Find the state of stress at points A, B, and C which are contained in a plane 180 in from the free end.

[b] Find internal forces and moments at the plane containing points A, B, and C.

\[ \sigma = \pm \frac{N}{A} \pm \frac{M_z}{I_z} y \pm \frac{M_y}{I_y} z \]
\[ \tau = \frac{V_y A}{I_y b_y} + \frac{V_z A}{I_z b_z} + \frac{M_x}{J_x} \]

\[ \sigma_A = \frac{N}{A} + \frac{M_z}{I_z} y + \frac{M_y}{I_y} z = 0 \]
\[ \tau_A = \frac{400 \left[ 6 \times 3 \times 3 \right]}{\frac{1}{12} (12^3 \times 6 \times 6)} + \frac{30 \left[ 12 \times 3 \times 15 \right]}{\frac{1}{12} (6^3 \times 12)} + 0 \]
\[ \tau_A = 833 \downarrow + 625 \leftarrow \beta_3 \]

Internal Forces and Moments:

\[ N = 4000 \text{ lb-in} \]
\[ M_z = 3000 \text{ lb-in} \]
\[ V_y = 30 \text{ lb} \]
\[ V_z = 40 \text{ lb} \downarrow \]
\[ M_x = 60 \text{ lb-in} \]
Point B \( (y = 0, z = -3 \text{ in}) \)

\[
\sigma_y = -\frac{M_y}{I_y} = -\frac{3000(-3)}{12(3^2)} = 41.66 \text{ psi};
\]

\[
\tau_{yz} = \frac{V_y G_y}{I_y t_y} + \frac{V_z G_z}{I_z t_z} + \frac{M_x}{I_z t_z} = 0 + 833 \text{ psi};
\]

\[
= 833 \text{ psi};
\]

\[
= \frac{60}{(246)(2x6)(2x3)} = 0 \text{ psi};
\]

\[
= 269 \text{ psi};
\]

\[
\alpha_B = \begin{bmatrix}
41.66 & -269 & 0 \\
-269 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \text{ psi};
\]

Point C \( (y = 6, z = -3) \)

\[
\sigma_x = \frac{4000(6)}{12(6^2)(6)} - \frac{3000(-3)}{12(3^2)(11)} = 69.4 \text{ psi};
\]

\[
\tau_{xy} = 0 + 0 + 0 = 0 \quad \text{because } M_y = 0 \text{ at the corner and shear stress due to } M_x \text{ is zero at corner}
\]

\[
\tau_{yz} = 0 + 0 + 0 = 0
\]

\[
\sigma_C = \begin{bmatrix}
69.4 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

when the stress components are acting only in one plane (e.g. x-y plane, y-z plane, or x-z plane) then we call that state of stress a **biaxial state of stress**

- Stress state at point A is not a biaxial state of stress, whereas the stress state at points B and C are biaxial state of stress. Therefore we may express the state of stress at these two points as

\[
\sigma_B = \begin{bmatrix}
41.66 & -269 \\
-269 & 0
\end{bmatrix} \text{ psi};
\]

\[
\sigma_C = \begin{bmatrix}
69.4 & 0 \\
0 & 0
\end{bmatrix} \text{ psi};
\]