Mechanics: The state of bodies at rest or motion

Mechanics

- rigid bodies
- deformable bodies
- fluid or gas

at rest

Mechanics of rigid bodies at rest = statics

at motion

Mechanics of rigid bodies at motion = dynamics

2.1 Scalars and Vectors

Physical quantities can be either scalar or vector quantities.

Scalar: quantity which has only magnitude (mass, volume, time)

Vector: quantity which has both magnitude and direction (force, velocity)

A vector is represented graphically by an arrow whose length represents its magnitude and its angle with the x-axis represents its direction.

\[ \vec{A} \]

vector \( \vec{A} \) is denoted by \( \vec{A} \) (where \( \rightarrow \) means \( \vec{A} \) is a vector)

Magnitude of vector \( \vec{A} \) is denoted by \( |\vec{A}| \)

1. Two vectors \( \vec{A} \) and \( \vec{B} \) are equal when they have the same magnitude and same angle.

\[ \vec{A} = \vec{B} \]

2. \( \vec{A} = \alpha \vec{B} \) (\( \alpha \) = scalar quantity) vector \( \vec{A} \) is the same as vector \( \vec{B} \) but \( |\vec{A}| = \alpha |\vec{B}| \)
Addition and Subtraction of Vectors (Graphically)

\[ \mathbf{R} = \mathbf{A} + \mathbf{B} \]

parallelogram

\[ \mathbf{R} = \mathbf{A} + \mathbf{B} \]

triangle

\[ \mathbf{RF} = \mathbf{C} + \mathbf{D} \]

\( \mathbf{c} \) and \( \mathbf{d} \) are called components and \( \mathbf{RF} \) is called resultant force.

two components may be combined together to give resultant. Resultant may be decompose to its component.

Finding Resultant by Trigonometry

Find magnitude and direction of the resultant of \( \mathbf{A} \) and \( \mathbf{B} \)

given both magnitude and direction of \( \mathbf{A} \) and \( \mathbf{B} \)

\[ \mathbf{A} \]

\[ \mathbf{B} \]

\[ \theta_A, \theta_B \]

\[ \beta = 180 - \theta_A + \theta_B \]

six quantities \((a, b, c, \alpha, \beta, \gamma)\), one may find 2 of them if four of them provided through sine law and cosine law.

\[
\begin{align*}
  c^2 &= a^2 + b^2 - 2ab \cos \gamma \\
  a^2 &= c^2 + b^2 - 2bc \cos \alpha \\
  b^2 &= c^2 + a^2 - 2ac \cos \beta
\end{align*}
\]

or

\[
\begin{align*}
  \sin \alpha &= \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}
\end{align*}
\]

Sine law
\[ \alpha = \frac{360 - 2(120)}{2} = 60 \]
\[ R = \sqrt{40^2 + 50^2 - 2(40)(50) \cos 60^\circ} \]
\[ R = 115.82 \text{ lb} \]
\[ \frac{\sin 60^\circ}{115.82} = \frac{\sin \beta}{50} \quad \Rightarrow \quad \beta = 70.9^\circ \]
\[ \theta = 120 - \beta + 22.6 + 90 \quad \Rightarrow \quad \theta = 161.2^\circ \]

**Resolving Resultant to Components**

**Rx** and **Ry** components of **R** along **x** and **y** axis respectively.

\[ \frac{R}{\sin 90^\circ} = \frac{R_y}{\sin \alpha} = \frac{R_y}{\cos \beta} \quad \Rightarrow \quad R_y = R \sin \beta \]
\[ \frac{R}{\sin 90^\circ} = \frac{R_x}{\sin \beta} = \frac{R_x}{\cos \alpha} \quad \Rightarrow \quad R_x = R \cos \alpha \]

\[ R_{x'} = R \cos \alpha \]
\[ R_{y'} = R \cos \beta \]

\[ \frac{R}{\sin [180 - (\pi + \beta)]} = \frac{R_{x'}}{\sin \beta} = \frac{R_{y'}}{\sin \alpha} \]
Given $F_{aa} = 30$
Find $F_{bb} = ?$ and $F = ?$

1. $F$ should be the diagonal of the parallelogram.
2. $F_{aa}$ and $F_{bb}$ should form sides of the parallelogram.
3. $F$, $F_{aa}$, $F_{bb}$ joined by their tails.

\[
\frac{F_{aa}}{\sin 80} = \frac{F}{\sin 40}
\]

\[
\frac{30}{\sin 80} = \frac{F}{\sin 40} \quad \Rightarrow \quad F = 19.58 \text{ lb}
\]

\[
\frac{30}{\sin 80} = \frac{F_{bb}}{\sin 60} \quad \Rightarrow \quad F_{bb} = 26.4 \text{ lb}
\]
In section 2.3 we have learned about the method of parallelogram in finding resultant or resolution of a force to its components along specified axis. This method seems to be complicated when resultant of more than two forces is required. Therefore, we need another technique to find resultant which will be described below.

Coplanar Forces: Are forces that lie in the same plane.

\[ \mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \text{ and } \mathbf{F}_4 \text{ are examples of coplanar forces.} \]

**Scalar Approach**

\[ \begin{align*}
\mathbf{F}_1 &= F_{1x} \mathbf{i} + F_{1y} \mathbf{j} \\
\mathbf{F}_2 &= F_{2x} \mathbf{i} + F_{2y} \mathbf{j} \\
\mathbf{F}_3 &= F_{3x} \mathbf{i} + F_{3y} \mathbf{j} \\
\mathbf{F}_4 &= F_{4x} \mathbf{i} + F_{4y} \mathbf{j}
\end{align*} \]

\[ \mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 \]

\[ \mathbf{R}_x = F_{Rx} \mathbf{i} + F_{Ry} \mathbf{j} \]

\[ F_{R} = \sqrt{F_{Rx}^2 + F_{Ry}^2} \]

\[ \theta = \tan^{-1} \left( \frac{F_{Ry}}{F_{Rx}} \right) \]

**Vector Approach**

\[ \mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 \]

\[ \mathbf{R} = \sum F_{Rx} \mathbf{i} + \sum F_{Ry} \mathbf{j} \]

\[ |\mathbf{R}| = \sqrt{F_{Rx}^2 + F_{Ry}^2} \]

\[ \theta = \tan^{-1} \left( \frac{F_{Ry}}{F_{Rx}} \right) \]
Using Sine Law Volume:
\[
\frac{600}{\sin 45} = \frac{AB}{\sin 60} \Rightarrow AB = 733 \text{ N}
\]
\[
\frac{600}{\sin 45} = \frac{CA}{\sin 75} \Rightarrow CA = 820 \text{ N}
\]

\[R + F = 900\]

\[F = 500\]

\[R_x = 400 \cos 20 + 500(\sin 20) - 600 \left( \frac{4}{5} \right) = 37.9\]

\[R_y = 500 \cos 20 + 600 \left( \frac{3}{5} \right) + 400 \sin 30 = 1029.8\]

\[R = \sqrt{37.9^2 + 1029.8^2} = 1031 \text{ N}\]

\[\theta = \tan^{-1} \left( \frac{1029.8}{37.9} \right) = 87.9^\circ\]
We are going to use the right-handed Cartesian coordinate system where when the right hand fingers are curled from \( x \) to \( y \), your thumb will point toward the \( z \)-direction.

Rectangular component of a vector in \( 3-D \)

Let \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) be unit vectors along \( x, y, z \) axes respectively.

\[ \mathbf{A} = (A_x, A_y, A_z) \]

Where:
- \( A_x \) = component of \( \mathbf{A} \) along the \( x \)-axis
- \( A_y \) = component of \( \mathbf{A} \) along the \( y \)-axis
- \( A_z \) = component of \( \mathbf{A} \) along the \( z \)-axis.

\[ \mathbf{A} = A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k} \quad (1) \]

One may think that vector \( \mathbf{A} \) is the sum of three vectors \( \mathbf{A}_x \), vector in the \( x-y \)-plane.

Unit vector: Any vector which has magnitude of unity \((=1)\).

Example: \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) each has a magnitude of one.

\[ |\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1 \]

Any vector can be made a unit vector if one divide itself by its magnitude.

\[ \mathbf{F} \Rightarrow \mathbf{F}_u = \frac{\mathbf{F}}{|\mathbf{F}|} \text{, unit vector of } \mathbf{F} \]

\[ \mathbf{B} \Rightarrow \mathbf{B}_u = \frac{\mathbf{B}}{|\mathbf{B}|} \text{, unit vector of } \mathbf{B} \]
\[ |\vec{A}| = |A| = \sqrt{A_x^2 + A_y^2 + A_z^2} \]

\[ F_x = |F| \cos \beta \]

Let \( \alpha \) = angle between \( \vec{A} \) and the \( x \)-axis,
\( \beta \) = angle between \( \vec{A} \) and the \( y \)-axis,
\( \gamma \) = angle between \( \vec{A} \) and the \( z \)-axis.

Projecting \( \vec{A} \) along \( x \)-axis: \( A_x = |A| \cos \alpha \)
Projecting \( \vec{A} \) along \( y \)-axis: \( A_y = |A| \cos \beta \)
Projecting \( \vec{A} \) along \( z \)-axis: \( A_z = |A| \cos \gamma \)

\[ \text{See Figure 2-28 page 43 (remember here projection) like components} \]

Substituting 2 into 1 yield:

\[ \vec{A} = |A| \cos \alpha \vec{i} + |A| \cos \beta \vec{j} + |A| \cos \gamma \vec{k} \]

\[ = |A| \left\{ \cos \alpha \vec{i} + \cos \beta \vec{j} + \cos \gamma \vec{k} \right\} \]

\[ \frac{\vec{A}}{|A|} = \left\{ \cos \alpha \vec{i} + \cos \beta \vec{j} + \cos \gamma \vec{k} \right\} = \vec{u}_A \]

\[ \Rightarrow 2 \]

\[ \vec{A} = |\vec{A}| \vec{u}_A, \quad \vec{u}_A = \cos \alpha \vec{i} + \cos \beta \vec{j} + \cos \gamma \vec{k} \]

\[ \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 \]

One needs only to define two angles out of three to determine direction of a vector.
Example: (1) Find the resultant of two forces in vector form
\[ \vec{F}_{R} = \vec{F}_1 + \vec{F}_2 \]
\[ \vec{F}_1 = 5\hat{i} + 4\hat{j} - 2\hat{k} \]
\[ \vec{F}_2 = 3\hat{i} + 4\hat{j} + 2\hat{k} \]
\[ \frac{\vec{F}_1 + \vec{F}_2}{F} = 8\hat{i} + 7\hat{j} \]
\[ \vec{F}_{R} = 8\hat{i} + 7\hat{j} \]

Let \[ \vec{C} = \vec{F}_1 - \vec{F}_2 = 2\hat{i} - 4\hat{k} \]

\( (\alpha, \beta, \gamma) \)

(2) Find the magnitude and direction of the above resultant
\[ |\vec{F}_R| = \sqrt{8^2 + 7^2} \]
\[ \vec{U}_{\vec{F}_R} = \vec{F}_R / |\vec{F}_R| = \frac{8\hat{i} + 7\hat{j}}{\sqrt{113}} = \frac{8}{\sqrt{113}} \hat{i} + \frac{7}{\sqrt{113}} \hat{j} \]
\[ \vec{U} = \cos \alpha \hat{i} + \sin \alpha \hat{j} \]

Direction:
\[ \cos \alpha = \frac{8}{\sqrt{113}} \Rightarrow \alpha = \cos^{-1} \left( \frac{8}{\sqrt{113}} \right) = 44.2^\circ \]
\[ \cos \beta = \frac{7}{\sqrt{113}} \Rightarrow \beta = \cos^{-1} \left( \frac{7}{\sqrt{113}} \right) = 48.8^\circ \]
\[ \gamma = 90^\circ \]
In previous lecture we talked about a vector or a force which can be written in general as:

\[ \vec{F} = |\vec{F}| \hat{u}, \quad \hat{u} = \cos \alpha \hat{i} + \cos \beta \hat{j} + \cos \gamma \hat{k} \]

What about if the angles were not provided explicitly, and instead we are given the coordinates of two points along which the force is acting.

Steps:
1. \( \vec{p}_{BA} = (x_b - x_a) \hat{i} + (y_b - y_a) \hat{j} + (z_b - z_a) \hat{k} \)
2. \( |\vec{p}_{BA}| = \sqrt{(x_b - x_a)^2 + (y_b - y_a)^2 + (z_b - z_a)^2} \)
3. \( \hat{u}_{BA} = \frac{\vec{p}_{BA}}{|\vec{p}_{BA}|} = \frac{1}{2} \)
4. \( \vec{F}_{BA} = \hat{u}_{BA} |\vec{F}| \)

Express \( F \) in vector form & determine its coordinate direction angles:

1. \( \vec{r}_{BA} = 4\hat{i} + 2.8\hat{j} - \hat{k} \)
2. \( |\vec{r}_{BA}| = \sqrt{4^2 + 2.8^2 + (-1)^2} = 4.94 \quad \frac{4.94}{0.553} \quad \frac{0.202}{1} \)
3. \( \hat{u} = \frac{\vec{r}_{BA}}{|\vec{r}_{BA}|} = 0.809 \hat{i} + \frac{2.76}{4.94} \hat{j} - \frac{1}{4.94} \hat{k} \)
4. \( F = 500 \hat{u} = 400 \hat{i} + 276 \hat{j} - 100 \hat{k} \)

\[ \hat{u} = \cos \alpha \hat{i} + \cos \beta \hat{j} + \cos \gamma \hat{k} \]

\[ \alpha = \cos^{-1}(0.809) = 36^\circ \]
\[ \beta = \cos^{-1}(0.553) = 56.5^\circ \]
\[ \gamma = \cos^{-1}(0.202) = 101.6^\circ \]
\[ F_B = |F_B| \hat{u}_{AB} = 520 \left\{ \frac{0 \hat{i} - 10 \hat{j} - 24 \hat{k}}{26} \right\} = 20 \hat{i} - 20 \hat{j} - 48 \hat{k} \]
\[ F_C = |F_C| \hat{u}_{AC} = 680 \left\{ \frac{16 \hat{i} + 18 \hat{j} - 24 \hat{k}}{34} \right\} = 320 \hat{i} + 360 \hat{j} - 480 \hat{k} \]
\[ F_D = |F_D| \hat{u}_{AD} = 560 \left\{ \frac{-12 \hat{i} + 8 \hat{j} - 24 \hat{k}}{28} \right\} = -240 \hat{i} + 160 \hat{j} - 480 \hat{k} \]

\[ \vec{R} = \vec{F_B} + \vec{F_C} + \vec{F_D} = 80 \hat{i} + 320 \hat{j} - 1440 \hat{k} \]
\[ |\vec{R}| = 1477.3 \]

\[ \vec{u}_R = 0.0541 \hat{i} + 0.2166 \hat{j} - 0.975 \hat{k} \]

\[ \alpha = \cos^{-1}(0.0541) = 86.9^\circ \]
\[ \beta = \cos^{-1}(0.2166) = 77.5^\circ \]
\[ \gamma = \cos^{-1}(0.975) = 12.84^\circ \]

\[ \text{Magnitude} \]

\[ \text{Direction} \]
Resultant of a system of forces (3-dimensions)

\[ F_1 = F_2 = F_3 = F_4 = F_5 = 100 \]

One needs to express each force in cartesian vector form:

\[ \vec{F} = |\vec{F}| \hat{u}_F \]

\[ \hat{u}_F = \cos \alpha \hat{i} + \cos \beta \hat{j} + \cos \gamma \hat{k} \]

\[ \vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k} \]

\[ \vec{F}_1 = 100 \hat{u}_F = 100 \left( \cos 60^\circ \hat{i} + \cos 60^\circ \hat{j} + \cos 135^\circ \hat{k} \right) \]

\[ F_2 = 100 \hat{u}_F = 100 \left( \frac{-2 \hat{i} + 3 \hat{j} + 2 \hat{k}}{\sqrt{4+9+4}} \right) = 100 \left( -0.485 \hat{i} + 0.726 \hat{j} + 0.485 \hat{k} \right) \]

\[ F_3 = \frac{3}{5} (100) \hat{j} + \frac{4}{5} (100) \hat{k} \]

\[ F_4 = 100 \hat{k} \]

\[ F_5 = (100 \cos 15^\circ) \cos 30^\circ \hat{i} - 100 \sin 15^\circ \hat{j} + (100 \cos 15^\circ) \cos 60^\circ \hat{k} \]
\[ \vec{R} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 + \vec{F}_5 \]

\[ = 100 (\cos 60 \cdot 0.485 + 0 + \cos 15 \cdot \cos 30) \hat{i} \]

\[ + 100 (\cos 60 \cdot 0.728 + \frac{3}{5} + 0 + \sin 15) \hat{j} \]

\[ + 100 (\cos 15 \cdot 0.485 + \frac{y}{5} + 1 + \cos 15 \cdot \cos 60) \hat{k} \]

\[ \vec{R} = 85.15 \hat{i} + 208.7 \hat{j} + 206.1 \hat{k} \]

\[ |\vec{R}| = \sqrt{(85.15)^2 + (208.7)^2 + (206.1)^2} = 305.4 \]

\[ \vec{R} = \frac{\vec{R}}{|\vec{R}|} \]

\[ \hat{\vec{R}} = 0.279 \hat{i} + 0.683 \hat{j} + 0.675 \hat{k} \]

\[ \hat{\vec{R}} = \cos \alpha \hat{i} + \cos \beta \hat{j} + \cos \gamma \hat{k} \]

\[ \alpha = \cos^{-1} (0.279) = 73.8^\circ \]

\[ \beta = \cos^{-1} (0.683) = 46.9^\circ \]

\[ \gamma = \cos^{-1} (0.675) = 47.55^\circ \]

\[ \vec{R} = R_x \hat{i} + R_y \hat{j} + R_z \hat{k} \]

\[ \vec{R} = \sum F_x \hat{i} + \sum F_y \hat{j} + \sum F_z \hat{k} \]
2.9 Dot Product

\[ \vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \]
\[ \vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k} \]
\[ \vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta \]
\[ \theta = \cos^{-1} \left( \frac{\vec{A} \cdot \vec{B}}{|\vec{A}| |\vec{B}|} \right) \]
\[ \theta = \cos^{-1} \left( \vec{u}_a \cdot \vec{u}_b \right) \]

\[ \vec{F}_{1-a} = \text{the projection of } \vec{F} \text{ along } a-a \]
\[ (\vec{F}_{1-a})_{a-a} = \vec{F} \cos \theta \]
\[ (\vec{F}_{1-a})_{a-a} = \vec{F} - \vec{u}_{a-a} \]
\[ (\vec{F}_{1-a})_{a-a} = \left[ \vec{F} - \vec{u}_{a-a} \right] \vec{u}_{a-a} \]

Find \( \theta = ? \)
\[ \vec{v}_{AB} = 2 \hat{i} - \hat{j} - 2 \hat{k} \]
\[ |\vec{v}_{AB}| = 3 \]
\[ \vec{u}_{AB} = \frac{2}{3} \hat{i} - \frac{1}{3} \hat{j} - \frac{2}{3} \hat{k} \]
\[ \vec{u}_y = -\hat{j} \]

\[ \cos \theta = \vec{u}_{AB} \cdot \vec{u}_y \]
\[ \theta = \cos^{-1} \left( \vec{u}_{AB} \cdot \vec{u}_y \right) = \cos^{-1} \left( \frac{1}{3} \right) = 70.5^\circ \]

The projection of \( AB \) along \( y \) axis: \( \vec{y} = (\vec{v}_{AB} \cdot \vec{u}_y) = -\hat{j} \)

The projection of \( AB \) along \( y \) axis in vector form: \( -\hat{j} \)
3.1

In this chapter we are going to apply what we have learned in chapter (2) to solve some real problems.

To maintain equilibrium in general, it is necessary to satisfy Newton's first law which states that if the resultant force acting on a body (or particle) is zero, then the body (or particle) is in equilibrium.

\[ \sum \vec{F} = \vec{0} \]

A vector has a magnitude zero if its individual components are equal to zeroes.

\[ \vec{F}_R = F_{Rx} \hat{i} + F_{Ry} \hat{j} + F_{Rz} \hat{k} = \vec{0} \hat{i} + \vec{0} \hat{j} + \vec{0} \hat{k} \]

\[ F_{Rx} = 0 \quad (\text{or } \sum F_x = 0) \]
\[ F_{Ry} = 0 \quad (\text{or } \sum F_y = 0) \]
\[ F_{Rz} = 0 \quad (\text{or } \sum F_z = 0) \]

3.2 The Free-Body Diagram (F.B.D)

FBD is a sketch of the particle after isolation from its surrounding showing all the acting forces (which are known) and the reactive forces (which are usually unknown).

![Free Body Diagram](image)
Forces on Cables (Cords, Ropes) & Spring.

cable can support only tension force and this force act in the direction of the cable and constant throughout its entire length.

spring can take tension or compression and its change in length is proportional to the force

\[ F \propto s \]

\[ F = kS = k(l - l_0) \]

\[ k = \text{stiffness of spring}. \]

\[ l - l_0 > 0 \quad (\text{tension}) \]
\[ l - l_0 < 0 \quad (\text{compression}) \]

Steps for Drawing FBD:

1. Draw the general shape of the particle after isolation
2. Indicate on the sketch all the active and reactive forces
3. Apply equations of equilibrium.

Ex.:

\[
\begin{align*}
F & \text{B.D. of spring} \\
F & \text{B.D. of pulley} \\
F & \text{B.D. of weight}
\end{align*}
\]
Particle A is subjected to three forces ($W$, $F_{spring}$, $400\,\text{lb}$).

Determine the necessary weight $W$ and the force in the spring to maintain the particle in equilibrium.

For particle A to be in equilibrium:

\[ \Sigma F_x = 0 \]
\[ \Sigma F_y = 0 \]  \quad \text{Approach I}

\[ \begin{align*}
\Sigma F_x &= W \left( \frac{4}{5} \right) + F_5 \sin 60° - 400 \sin 30° = 0 \quad \text{(1)} \\
\Sigma F_y &= -W \left( \frac{3}{5} \right) - F_5 \cos 60° + 400 \cos 30° = 0 \quad \text{(2)}
\end{align*} \]

Solving (1) & (2) \quad W = 435\,\text{lb} \quad \text{&} \quad F_5 = 171\,\text{lb}

\text{Approach II: Resultant:}

\[ \begin{align*}
F_5 &= -400 \sin 30° \hat{i} + 400 \cos 30° \hat{j} \\
\vec{W} &= \frac{4}{5} W \hat{i} - \frac{3}{5} W \hat{j} \\
\vec{F}_5 &= -F_5 \sin 60° \hat{i} - F_5 \cos 60° \hat{j}
\end{align*} \]

\[ \vec{R} = \vec{F}_5 + \vec{W} + \vec{F}_5 = \left( \frac{4}{5} W - F_5 \sin 60° - 400 \sin 30° \right) \hat{i} + \left( -W \left( \frac{3}{5} \right) - F_5 \cos 60° - 400 \cos 30° \right) \hat{j} \]

For equilibrium, \[ \vec{R} = 0 \quad \left( R_x = 0, \quad R_y = 0 \right) \]

\[ \begin{align*}
\frac{4}{5} W - F_5 \sin 60° - 400 \sin 30° &= 0 \quad (R_x = 0) \quad \text{(3)} \\
-\frac{3}{5} W - F_5 \cos 60° - 400 \cos 30° &= 0 \quad (R_y = 0) \quad \text{(4)}
\end{align*} \]

Eq. (3) & (4) are same as \[ \text{Eq. 3 & 4} \]

\[ \begin{align*}
W &= 435\,\text{lb} \quad \text{&} \quad F_5 = 171\,\text{lb}
\end{align*} \]

\text{Approach I: Scalar approach} \\
\text{Approach II: Vector approach}
3-42. Determine the magnitudes of \( F_1 \), \( F_2 \), and \( F_3 \) for equilibrium of the particle.

\[
\begin{align*}
\vec{F}_1 &= |F_1| \left( \cos 60 \hat{i} + \cos 60 \hat{j} + \cos 15 \hat{k} \right) \\
\vec{F}_2 &= |F_2| \hat{k} \\
\vec{F}_3 &= -|F_3| \hat{j} \\
\vec{F}_4 &= (150 \cos 45^\circ) \cos 30 \hat{i} - (150 \cos 45^\circ) \sin 30 \hat{j} + 150 \sin 45 \hat{k} \\
\vec{F}_5 &= -225 \hat{i} \\
\vec{R} &= \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 + \vec{F}_5
\end{align*}
\]

\[
\begin{align*}
\vec{R} &= \left( |F_1| \cos 60 + 0 + 0 + (150 \cos 45^\circ) \cos 30 - 225 \right) \hat{i} \\
&\quad + \left( |F_1| \cos 60 + 0 - |F_3| - (150 \cos 45^\circ) \cos 30 \right) \hat{j} \\
&\quad + \left( |F_1| \cos 135 + |F_2| + 0 + 150 \sin 45 \right) \hat{k}
\end{align*}
\]

For equilibrium \( \vec{R} = \vec{0} = 0 \hat{i} + 0 \hat{j} + 0 \hat{k} \)

\[
\begin{align*}
\sum F_x &= 0 \quad \Rightarrow \quad |F_1| \cos 60 + (150 \cos 45^\circ) \cos 30 - 225 \Rightarrow |F_1| = 266 \text{ lb} \\
\sum F_y &= 0 \quad \Rightarrow \quad |F_1| \cos 60 - |F_3| - (150 \cos 45^\circ) \cos 30 = 0 \Rightarrow |F_3| = 80.4 \text{ lb} \\
\sum F_z &= 0 \quad \Rightarrow \quad |F_1| \cos 135 + |F_2| + 150 \sin 45 = 0 \Rightarrow |F_2| = 82.2 \text{ lb}
\end{align*}
\]
3-49. Determine the force in each cable needed to support the 500-lb cylinder.

\[
\begin{align*}
A &= (0, 0, 0) \\
B &= (-3, 6, 6) \\
C &= (-2, 3, 6) \\
D &= (5, 0, 0)
\end{align*}
\]

\[
\vec{\ell}_{AB} = -0.333\hat{i} - 0.666\hat{j} + 0.666\hat{k}
\]

\[
\vec{\ell}_{AC} = -0.285\hat{i} + 0.4286\hat{j} + 0.857\hat{k}
\]

\[
\vec{\ell}_{AD} = \hat{z}
\]

\[
\vec{F}_{AB} = -0.333\left|\vec{\ell}_{AB}\right|\hat{i} - 0.666\left|\vec{\ell}_{AB}\right|\hat{j} + 0.666\left|\vec{\ell}_{AB}\right|\hat{k}
\]

\[
\left|\vec{\ell}_{AC}\right| = -0.285\hat{i} + 0.4286\hat{j} + 0.857\hat{k}
\]

\[
\vec{F}_{AC} = \left|\vec{\ell}_{AC}\right|\hat{i} + 0.4286\left|\vec{\ell}_{AC}\right|\hat{j} + 0.857\left|\vec{\ell}_{AC}\right|\hat{k}
\]

\[
\vec{F}_{AD} = \hat{z}
\]

\[
\vec{R} = \vec{F}_{AB} + \left|\vec{\ell}_{AC}\right| + \vec{F}_{AD} + \vec{W}
\]

\[
\vec{R} = \begin{bmatrix}
-0.333\left|\vec{\ell}_{AB}\right| & -0.285\left|\vec{\ell}_{AC}\right| & -0.666\left|\vec{\ell}_{AB}\right| + \hat{z} \\
-0.666\left|\vec{\ell}_{AB}\right| & + 0.4286\left|\vec{\ell}_{AC}\right| & -0.857\left|\vec{\ell}_{AC}\right| - 500 \hat{k}
\end{bmatrix}
\]

For equilibrium: \( \vec{R} = 0\hat{i} + 0\hat{j} + 0\hat{k} \)

\[
-0.333\left|\vec{\ell}_{AB}\right| - 0.285\left|\vec{\ell}_{AC}\right| + \left|\vec{\ell}_{AD}\right| = 0 \tag{1}
\]

\[
-0.666\left|\vec{\ell}_{AB}\right| + 0.4286\left|\vec{\ell}_{AC}\right| = 0 \tag{2}
\]

\[
+0.666\left|\vec{\ell}_{AB}\right| + 0.857\left|\vec{\ell}_{AC}\right| - 500 = 0 \tag{3}
\]

(2) + (3) \Rightarrow \left|\vec{\ell}_{AC}\right| = 389 \text{ lb}

Substitute into (2) or (3) \Rightarrow \left|\vec{\ell}_{AB}\right| = 250 \text{ lb}

Substitute into (1) \Rightarrow F_{AB} = 194 \text{ lb}
3-73. Determine the magnitude of $\mathbf{P}$ and the coordinate direction angles of the 200-lb force required for equilibrium of the particle. Note that $\mathbf{F}_3$ acts in the octant shown.

\[ \begin{align*}
\mathbf{F}_1 &= 360 \left\{ \frac{1}{\sqrt{66}} \mathbf{i} - \frac{7}{\sqrt{66}} \mathbf{j} + \frac{5}{\sqrt{66}} \mathbf{k} \right\} \\
\mathbf{F}_2 &= -44.3 \mathbf{i} - 310 \mathbf{j} + 171 \mathbf{k} \\
\mathbf{F}_3 &= +0 \mathbf{i} - 120 \mathbf{j} + 0 \mathbf{k} \\
\mathbf{P} &= 0 \mathbf{i} + P \cos 20^\circ \mathbf{j} + P \sin 20^\circ \mathbf{k} \\
\mathbf{F}_4 &= -300 \mathbf{k} \\
\Sigma F_x &= 0 \quad -44.3 + F_{3x} = 0 \quad (1) \\
\Sigma F_y &= 0 \quad -310 - 120 - F_{3y} + P \cos 20^\circ = 0 \quad (2) \\
\Sigma F_z &= 0 \quad 171 - F_{3z} + P \sin 20^\circ - 300 = 0 \quad (3)
\end{align*} \]

From (1) \quad \begin{align*}
F_{3x} &= 44.3 \\
\end{align*}

From (2) \quad \begin{align*}
F_{3y} &= P \cos 20^\circ - 430 \\
\end{align*}

From (3) \quad \begin{align*}
F_{3z} &= P \sin 20^\circ - 123 \\
\end{align*}

But \quad \begin{align*}
& \left( \frac{1}{15} \right)^2 = \frac{F_{3x}^2}{F_{3x}^2} + \frac{F_{3y}^2}{F_{3y}^2} + \frac{F_{3z}^2}{F_{3z}^2} \\
& \left(2000\right)^2 = (44.3)^2 + \left( P \cos 20^\circ - 430 \right)^2 + \left( P \sin 20^\circ - 123 \right)^2 \\
\end{align*}

Solve for \quad P = 639 \text{ lb}

Substituting into (2) \& (3) \quad \begin{align*}
& F_{3y} = 170.5 \\
& F_{3z} = 955 \text{ lb}
\end{align*}

\[ \text{how can you find } \alpha, \beta, \gamma \text{ for } \frac{\mathbf{F}}{\mathbf{F}} = \mathbf{321} \]
Ch. 4: Force System Resultants

1. Moment of a force - Scalar formulation

\[ M = Fd \ (magnitude \ \text{direction}) \]

where \( d \) is the shortest distance between \( F \) and the point.

2. Moment of a force - Vector formulation

\[ \vec{M} = \vec{r} \times \vec{F} = \begin{vmatrix} i & j & k \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} = (z F_x - y F_z) \hat{i} - (z F_y - x F_z) \hat{j} + (y F_x - x F_y) \hat{k} \]

\( \vec{r} \) = the position vector from the point you want to calculate the moment about to any point in the line of action of the force.

Magnitude = \( |\vec{M}| = |\vec{r} \times \vec{F}| = |\vec{r}| |\vec{F}| \sin \theta = Fd \quad d = \text{shortest distance} \)

3. The moment of a set of forces equal to the sum of the moments of the individual force or the moment of their resultant

\[ M = \sum (x_i \vec{F}_i + y_i \vec{F}_i + z_i \vec{F}_i) = \vec{r} \times \vec{F} \]
Determine $F_A$ such that the total moment about $C$ equals to zero.

\[ \frac{3}{5} F_A - 30 \sin 60^\circ \cdot 6 = 0 \]

\[ F_A = \frac{28.9}{100} \text{ lb} \]

Determine the maximum and minimum moment of $F$ about point $A$ (specified $0 \leq \theta \leq 90^\circ$).

1. Max. moment occurs when the force has the largest distance from $A$, which means $F \perp AC$.

   \[ \theta = \sin^{-1} \left( \frac{3}{5} \right) \Rightarrow \theta = 56.3^\circ \]

   \[ M = Fd = 400 \left( \frac{3^2 + 1^2}{\sqrt{3^2 + 1^2}} \right) = 1442.4 \text{ N.m} \]

2. Min. moment occurs when the force $F$ passes through $A$.

   \[ \theta = 0^\circ + 90^\circ = 56.3^\circ + 90^\circ = 146.3^\circ \]

   \[ M = 0 \]

Determine the moment of $F$ about $A$ and the shortest distance.

\[ \overrightarrow{M_A} = \overrightarrow{Y_A} \times \overrightarrow{F} = \begin{bmatrix} 4 & 3 & 0 \\ 338.87 & 426.5 & -365.9 \end{bmatrix} \]

\[ |\overrightarrow{M_A}| = \sqrt{1098^2 + 1464^2 + (845)^2} = 2016 \text{ N.m} \]

\[ F = 500 \text{ N} \]

\[ |\overrightarrow{Ma}| = F \cdot d = 500 \cdot d \Rightarrow d = \frac{|\overrightarrow{Ma}|}{F} = \frac{2016}{500} = 4.03 \text{ m} \]
4.5 Moment of a Force about a specified Axis

Moment of $F$ about $x$-axis $= 50 \overrightarrow{2i}$ N-m
Moment of $F$ about $y$-axis $= 0$ (intersects the $y$-axis)
Moment of $F$ about $z$-axis $= 0$ (parallel to the $z$-axis)

Moment of $F_2$ about $z$-axis $= 20 \overrightarrow{3k}$ N-m
Moment of $F_3$ about $y$-axis $= 0$ (intersects the $y$-axis)

Moment of $F_4$ about $z$-axis $= 0$ (parallel to the $z$-axis)

$$M_{\text{ax}} = F(\perp \text{distance} = d)$$

$$M_{\text{ax}} = \vec{u}_{\text{ax}} \cdot (\vec{r} \times \vec{F})$$

$$\vec{u}_{\text{ax}} = \begin{bmatrix} u_x & u_y & u_z \end{bmatrix} = \begin{bmatrix} 2 \overrightarrow{i} & \overrightarrow{0} & \overrightarrow{0} \end{bmatrix}$$

$$\vec{r} \times \vec{F} = \begin{bmatrix} F_x & F_y & F_z \end{bmatrix} = \begin{bmatrix} 3 \overrightarrow{i} & 2 \overrightarrow{j} & 2 \overrightarrow{k} \end{bmatrix}$$

$$\overrightarrow{F} = [20\overrightarrow{i} + 40\overrightarrow{j} + 20\overrightarrow{k}] \text{ W}$$

$$\overrightarrow{r} = (-3, 3, 2)$$

$$\vec{u}_{\text{ax}} = \frac{1}{\sqrt{2}} \overrightarrow{i} + \frac{1}{\sqrt{2}} \overrightarrow{j}$$

$$M_{\text{ax}} = \vec{u}_{\text{ax}} \cdot (\vec{r} \times \vec{F}) = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} - 3 = -2$$

$$\frac{1}{\sqrt{2}} \times 3 + \frac{1}{\sqrt{2}} \times (20) = \frac{1}{\sqrt{2}} (310) = 219 \text{ N-m}$$
A couple is two forces equal in magnitude, parallel, and opposite in directions.

Moment of a couple is a free vector (i.e. independent of the point you choose).

\[ \mathbf{M}_c = \mathbf{F} \cdot \mathbf{d} \]

\[ \mathbf{M} = \mathbf{r} \times \mathbf{F}_1 \neq (\mathbf{r} \times \mathbf{F}_2) \quad [\mathbf{F}_1 \neq \mathbf{F}_2] \]

\( \mathbf{r} = \) position vector going from one face to the other
\( \mathbf{F} = \) the force where the position vector is going to \( (\mathbf{F}_1 \neq \mathbf{F}_2) \)

\[
\begin{align*}
\mathbf{F}_1 &= 50 \mathbf{i} - 20 \mathbf{j} + 80 \mathbf{k} \\
\mathbf{F}_2 &= -50 \mathbf{i} + 20 \mathbf{j} - 80 \mathbf{k} \\
\mathbf{F} &= 4 \mathbf{i} - 8 \mathbf{j} + 14 \mathbf{k} \\
\mathbf{r} &= \mathbf{i} - \mathbf{j} + \mathbf{k} \\
\mathbf{M}_c &= \mathbf{r} \times \mathbf{F}_1 = 4 \mathbf{i} - 8 \mathbf{j} + 14 \mathbf{k} \\
&= 50 \mathbf{i} - 20 \mathbf{j} + 80 \mathbf{k} \\
&= \left\langle 360 \mathbf{i} + 380 \mathbf{j} + 320 \mathbf{k} \right\rangle
\end{align*}
\]

4.86

B(0, 3, 8)

A(-4, 5, -6)
When a force is moved from one point A to another point B, it will have the same force at B (same magnitude + direction) in addition to a couple moment at B = Fd.

Eg: Replace the system of forces and couple by a single force and couple moment at point O.

\[
\begin{align*}
&\text{100N} \\
&\text{200N} \\
&\text{400N} \\
\end{align*}
\]

\[
\frac{41.10^\circ}{174}
\]

\[
\text{Single force:} \quad \begin{align*}
F_x &= ZF_x = -60 - 60 \cos 30^\circ = -51.96 \\
F_y &= ZF_y = -140 - 60 \sin 30^\circ = -170 \\
R &= \sqrt{(51.96)^2 + 170^2} = 178 N \\
\theta &= \tan^{-1} \left( \frac{178}{51.96} \right) = 74^\circ
\end{align*}
\]

\[
\begin{align*}
M &= 40 + [60 \sin 30^\circ] \cdot 4 + [60 \cos 30^\circ] \cdot 8 \\
&\quad + 140(15) \\
M &= 40 + 120 + 416 + 2100 \\
&= 2676 \text{ N-m} \\
&= 2.68 \text{ kN-m}
\end{align*}
\]
4.8 Resultants of a force and couple system

\[ \vec{F}_R = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 \]

\[ \vec{M}_C = \vec{M}_{C1} + \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 \]

\[ \begin{align*}
\vec{F}_1 &= 360i + 480j - 100k \\
\vec{F}_2 &= 100i - 100j - 50k \\
\vec{F}_3 &= -500k \\
\vec{r}_1 &= 12k \\
\vec{r}_2 &= 12k \\
\vec{r}_3 &= 12k \\
\end{align*} \]

\[ \begin{align*}
\vec{F}_R &= \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = (460i + 300j - 650k) \text{ N} \\
\vec{M}_C &= \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \vec{r}_3 \times \vec{F}_3 = \vec{r}_1 \times (\vec{F}_1 + \vec{F}_2) + \vec{r}_3 \times \vec{F}_3 \\
\end{align*} \]

\[ \begin{vmatrix}
i & j & k \\0 & 0 & 12 \\400 & 300 & -150 \end{vmatrix} = 0 \]

\[ \begin{vmatrix}
i & j & k \\0 & 0 & -1 \\0 & 0 & -500 \end{vmatrix} = 0 \]

\[ \begin{align*}
\vec{F}_R &= \{ -3600i + 4800j \} + \{ -3100i + 4800j \} = \{ -3100i + 4800j \} \text{ N} \\
\end{align*} \]
4.10 Reduction of a simple Distributed loading

Two systems are equivalent if they produce the same force and moment about a given point.

\[ F = \int a(x) \, dx \]

1. \( F = \text{area under curve} \) \( \Rightarrow \)

\[ \int x \cdot a(x) \, dx = F \, d \] \( \Rightarrow \)

\[ d = \frac{\int x \cdot a(x) \, dx}{\int a(x) \, dx} \]

\[ a(x) = \frac{b}{2} x^2 \]

\[ F = \int_{a}^{b} \frac{b}{2} x^2 \, dx = \frac{b}{2} \int_{a}^{b} x^2 \, dx = \frac{b}{2} \left[ \frac{x^3}{3} \right]_{a}^{b} = \frac{1}{3} b a q \]

\[ d = \frac{\int x \cdot \frac{b}{2} x^2 \, dx}{\int \frac{b}{2} x^2 \, dx} = \frac{\frac{b}{2} \int x^3 \, dx}{\frac{b}{2} \int x^2 \, dx} = \frac{\frac{1}{4} b a^2}{\frac{1}{3} b a} = \frac{3}{4} a \]

\[ a = \frac{b}{2} \]

\[ d = \frac{ab}{1-b/2} \]

\[ h = \frac{1}{2} ah \]

\[ d = \frac{10}{3} \]

\[ F = 20 + 15 = 35 \]

\[ M_A = -15 \cdot 2 - 20 \cdot 4 = -F \cdot d = -35 \, d \] \( \Rightarrow \)

\[ d = \frac{110}{35} = 3.14 \, m \]
Determine the force in each member of the truss. State if the force is tension (T) or compression.

No need to find support reaction as we can start with a joint that has two knowns. (Joint A)

1. **Joint A**

   \[ \sum F_y = 0 : 130 \left( \frac{12}{13} \right) + AC \left( \frac{4}{5} \right) = 0 \]
   \[ \sum F_x = 0 : AB + AC \left( \frac{3}{5} \right) - 130 \left( \frac{5}{13} \right) = 0 \]
   \[ AC = 150 \text{ lb} \]
   \[ AB = 140 \text{ lb} \]

2. **Joint B**

   \[ \sum F_y = 0 : BC = 0 \]
   \[ \sum F_x = 0 : BD - BA = 0 \]
   \[ BD - 140 = 0 \Rightarrow BD = 140 \text{ lb} \]

3. **Joint C**

   \[ \sum F_y = 0 : CA \left( \frac{4}{5} \right) + CD \left( \frac{4}{5} \right) = 0 \Rightarrow CD = -CA \]
   \[ CD = -150 \text{ lb} \]
   \[ \sum F_x = 0 : CE + \left( \frac{3}{5} \right) CD - CA \left( \frac{3}{5} \right) = 0 \]
   \[ CE = -180 \text{ lb} \]

4. **Joint D**

   \[ \sum F_y = 0 : DF - DB - CD \left( \frac{3}{5} \right) = 0 \Rightarrow DF = 230 \text{ lb} \]
   \[ \sum F_x = 0 : DE + DC \left( \frac{4}{5} \right) = 0 \Rightarrow DE = -120 \text{ lb} \]

5. **Joint E**

   \[ \sum F_x = 0 : EF \left( \frac{3}{5} \right) - EC = 0 \]
   \[ EF = -300 \text{ lb} \]
<table>
<thead>
<tr>
<th>Member</th>
<th>Force (lb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB</td>
<td>140 (T)</td>
</tr>
<tr>
<td>AC</td>
<td>150 (C)</td>
</tr>
<tr>
<td>CB</td>
<td>0</td>
</tr>
<tr>
<td>BD</td>
<td>140 (T)</td>
</tr>
<tr>
<td>DF</td>
<td>230 (T)</td>
</tr>
<tr>
<td>CE</td>
<td>180 (C)</td>
</tr>
<tr>
<td>CD</td>
<td>150 (T)</td>
</tr>
<tr>
<td>ED</td>
<td>120 (C)</td>
</tr>
<tr>
<td>EF</td>
<td>300 (C)</td>
</tr>
</tbody>
</table>

\[
\frac{G-33}{279}
\]

Consider section \( \overline{CA} \)

\[ \sum F_y = 0 \quad (\text{solve for BH}) \]

\[ \sum F_x = 0 \quad (\text{solve for BC}) \]

\( \sum F_y = 0 \quad (\text{solve for HC}) \]

Consider section \( \overline{CA} \)

\[ \sum F_y = 0 \quad 40 + 60 + 80 - HB \frac{1}{2} = 0 \quad HB = 255 \text{ lb} \]

\[ \sum M_{HC} = 0 \quad 40(3.5) + 60(2) = BC(2) \quad BC = 130 \text{ lb} \]

Consider section \( \overline{C \bar{B}} \)

\[ \sum F_y = 0 \quad 40 + 60 + 80 + CH = 0 \]

\[ CH = 180 \text{ c} \]
(1) Draw free body diagram of the whole structure, showing all the applied loads and reactions.

(2) Count number of external unknown reactions and compare them with the total number of equations of equilibrium (normally three questions) if number of reactions more than number of equations (greater than 3) then one need to disconnect members. Otherwise apply equations of equilibrium and find the unknowns.

(3) Before disconnecting, if there is pulley attached to the frame, disconnect the pulley and find its pin reactions and put these reactions on the structure where the pulley was connected as applied load in the opposite direction.

(4) Disconnect members and draw F.B.D of each member and of any joint which has load applied to it. Start naming reactions with the two force members (remember disconnection means equal forces and opposite direction).

(5) Apply again equations of equilibrium to members and joints separately. Remember there are three equations of equilibrium for each member (a non two-force member) and two equations of equilibrium for each joint.

If total number of equations of equilibrium equal to the total number of unknowns then proceeding to solve for the unknown, otherwise you must have made a mistake.

Note: When two members meet at a joint where load is acting then their forces are independent and the relationship will come through the equilibrium of the joint.