How to convert a differential equation model to a transfer function?

Problem:

Given a linear differential equation that describes the relationship between the input variable u(t) and the output variable y(t), determine the corresponding transfer function.

Procedure :

1. Apply Laplace transform assuming zero initial conditions.
2. The trasfer function G(s) is obtained as

$$G(s) = \frac{L\{y(t)\}}{L\{u(t)\}}$$

Example

Find the transfer function of the system described by the differential equation $2\ddot{y} + 3\dot{y} + 4y = 0.5\dot{u} + u$

where $u(t)$ is the input and $y(t)$ is the output.

Solution :

$$2\left(s^2Y(s) - sy(0) - \dot{y}(0)\right) + 3\left(sY(s) - y(0)\right) + 4Y(s)$$
$$= 0.5\left(sU(s) - u(0)\right) + U(s)$$

Assume all initial conditions are zeros.

$$2s^2Y(s) + 3sY(s) + 4Y(s) = 0.5sU(s) + U(s)$$

$$(2s^2 + 3s + 4)Y(s) = (0.5s + 1)U(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{0.5s + 1}{2s^2 + 3s + 4}$$