9.2 Matrix Solution of Linear Systems

Objectives

Learn how to solve systems of m linear equations of n variables, using Gauss & Gauss-Jordan Methods

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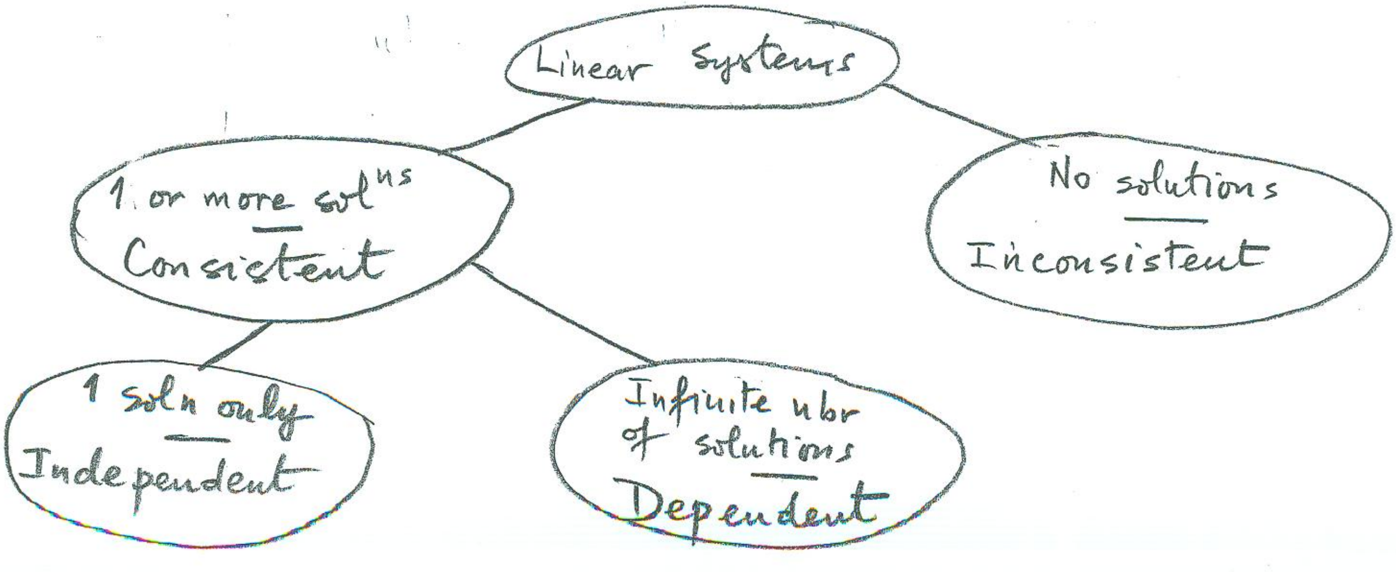
A linear systems of m equations & n variables is a system of the form

$$(S)\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = c_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = c_2 \\ \vdots \qquad\qquad\qquad\qquad \vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = c_n \end{cases}$$

A solution is an n-tuple $(x_1, x_2, \ldots, x_n)$ which satisfy all the equations.

Similarly to the case of linear system of 2 variabl. we have the following possibilities



Augmented Matrix.

The augmented matrix of a system (s)

$$\begin{cases} a_{11} x_1 + a_{22} x_2 + \cdots + a_{1n} x_n = c_1 \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = c_2 \\ \vdots \\ a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n = c_n \end{cases}$$

is

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & c_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & c_2 \\ \vdots & \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} & c_n \end{array}\right]$$

Exp1. The augmented matrix of

$$\begin{cases} 2x - y + 2z = 3 \\ 4x + 2y - z = 2 \\ \quad 3y + 4z = -1 \end{cases}$$

is

$$\left[\begin{array}{ccc|c} 2 & -1 & 2 & 3 \\ 4 & 2 & -1 & 2 \\ 0 & 3 & 4 & -1 \end{array}\right]$$

Exp2. The system of the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ -2 & 3 & -1 & 2 \\ 4 & 0 & 3 & 7 \end{array}\right]$$

is

$$\begin{cases} x - 3y + 2z = -1 \\ -2x + 3y - z = 2 \\ 4x + 3z = 7 \end{cases}$$

Special Easy systems.

1)
$$\begin{cases} x - 2y + 3z = 1 & (1) \\ y - 2z = 2 & (2) \\ z = 3 & (3) \end{cases}$$

$(3) \Rightarrow z = 3$

$\overset{(2)}{\Longrightarrow} y = 2 + 2z = 2 + 2(3) = 8$

$\overset{(1)}{\Longrightarrow} x = 1 + 2y - 3z$
$= 1 + 16 - 9 = 8$

$\Rightarrow (8, 8, 3)$ is the solution.

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

is an example of <u>row-echelon form.</u>

An augmented matrix is in echelon form if

1) The first nonzero nbr in any row is 1.
2) Under the first nonzero number in any row is a column of zero.
3) rows of zeros are at the end.

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2) <u>Reduced-Row Echelon form</u>,

$$\begin{cases} x = 4 \\ y = 5 \\ z = -1 \end{cases}$$

$\Rightarrow (x, y, z) = (4, 5, 1)$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

is an example of Reduced-Row-Echelon-form.

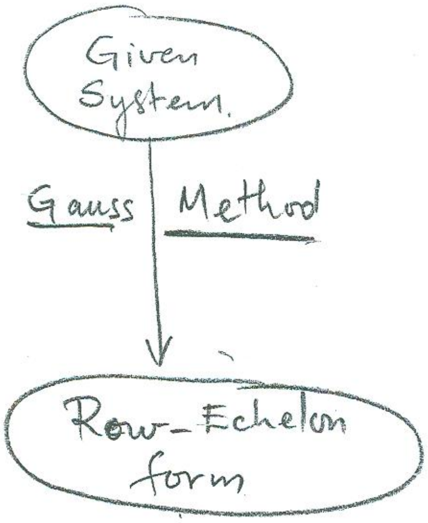
An augmented matrix is in reduced row-echelon form if it is of the form

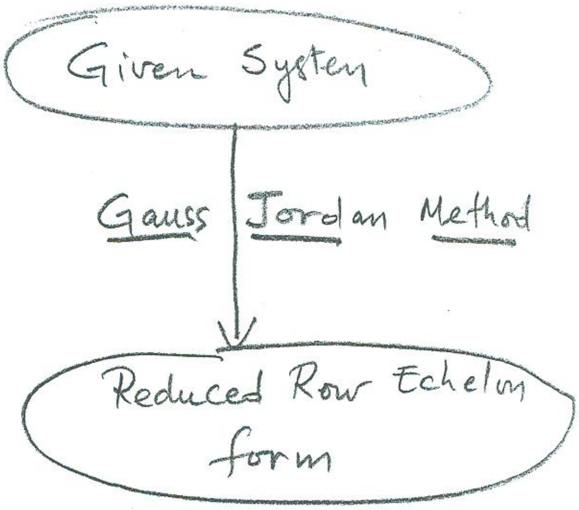
$$\left(\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & * \\ 0 & 1 & & \vdots & * \\ \vdots & & \ddots & & \vdots \\ 0 & & & 1 & * \end{array}\right)$$

<u>Elementary row operations that gives equivalent systems</u>

1. Interchanging two rows
2. Multiplying a row by a nonzero nbr.
3. Adding a multiple of a row to another row.

<u>Gauss & Gauss Jordan Elimination Methods.</u>





$$\left(\begin{array}{cccc|c} 1 & \times & & \times & \times \\ 0 & 1 & & \times & \times \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & & & 1 & \times \end{array}\right) \qquad \left(\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & \times \\ 0 & 1 & & \vdots & * \\ \vdots & & \ddots & 0 & \\ 0 & \cdots & 0 & 1 & \times \end{array}\right)$$

Exp3. Let's illustrate the Gauss Method to solve

$$\begin{cases} 2x - y + z = 6 \\ x + y + 2z = 3 \\ -x + 3y = -5 \end{cases}$$

Step 1. put the augmented matrix

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 6 \\ 1 & 1 & 2 & 3 \\ -1 & 3 & 0 & -5 \end{array}\right]$$

Step 2 Use elementary row operations to get row-echelon form $\left[\begin{array}{cccc} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right]$

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 6 \\ 1 & 1 & 2 & 3 \\ -1 & 3 & 0 & -5 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 2 & -1 & 1 & 6 \\ -1 & 3 & 0 & -5 \end{array}\right]$$

$$\xrightarrow[R_1 + R_3]{-2R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & -3 & -3 & 0 \\ 0 & 4 & 2 & -2 \end{array}\right] \xrightarrow{\frac{R_2}{-3}} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 4 & 2 & -2 \end{array}\right]$$

$$\xrightarrow{-4R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & -2 \end{array}\right] \xrightarrow{\frac{R_3}{-2}} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

We got the row-echelon form.

Step 3. Write system of last matrix & do Back substitution.

$$\begin{cases} x + y + 2z = 3 & (1') \\ \quad\ \ y + z = 0 & (2') \\ \qquad\quad z = 1 & (3') \end{cases}$$

Substitute $(3')$ in $(2')$

$$y + 1 = 0 \Rightarrow \boxed{y = -1}$$

Back substitute $z = 1$, $y = -1$ in $(1')$

$$x + (-1) + 2(1) = 3$$

$$x + 1 = 3 \Rightarrow \boxed{x = +2}$$

$\Rightarrow$ The sol^n is $\boxed{(2, -1, 1)}$

Exp3. Let's show the Gauss Jordan Method on same example

Step 1 same

Step 2 we want to get $\begin{pmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}$

We do same steps until $\begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

We continue now

$$\begin{pmatrix} 1 & 1 & \boxed{2} & 3 \\ 0 & 1 & \boxed{1} & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow[-R_3+R_2]{-2R_3+R_1} \left(\begin{array}{ccc|c} 1 & \boxed{1} & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{-R_2+R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Step 3

$$\begin{cases} x = 2 \\ y = -1 \\ z = 3 \end{cases} \rightarrow \boxed{(x, y, z) = (2, -1, 3)}$$

Dependent & Inconsistent Systems.

In the process of G.E.M or G.J.E.M

1) If we get a row of the form $[0 \ 0 \quad a]$ & $a \neq 0$ $\Rightarrow$ The system is inconsistent (No sol^n)

2) A row of zeros should be deleted.

2') If two rows are equal one row can be deleted.

3) If we get the row echelon form. (zero rows deleted)

   a) Nbr(eq^n) = Nbr(var) $\Rightarrow$ Independent (1 sol)

   b) Nbr(eq^n) < Nbr(var) $\Rightarrow$ Dependent

   & Nbr(free variables) = Nbr(var) - Nbr(eq^n).

Exp 5 Solve the system
$$\begin{cases} x - 2y - 2z = 1 \\ x + y + z = 2 \\ x + 2y + 2z = 1 \end{cases}$$

Exp 6. Solve
$$\begin{cases} x + 2y = 1 \\ x + 3y + z = 4 \\ 2y + 2z = 6 \end{cases}$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 3 & 1 & 4 \\ 0 & 2 & 2 & 6 \end{bmatrix} \xrightarrow{-R_1 + R_2} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 2 & 6 \end{bmatrix} \xrightarrow{-2R_2 + R_3}$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

$$\begin{cases} x + 2y = 1 & (1') \\ \quad y + z = 3 & (2') \end{cases} \rightarrow \text{Row echelon form}$$

2 eq^{ns}, 3 variables & row-echelon form $\Rightarrow$ Dependent

Take last eq^n with 2 variables, $y + z = 3$

Solve for y, $y = 3 - z$.

If $z = c$ anbr $\Rightarrow y = 3 - c$

$\overset{(1')}{\Rightarrow}$ $x = 1 - 2y = 1 - 2(3 - c) = 2c - 5$

$\Rightarrow SS = \{(2c - 5, 3 - c, c) : c \in \mathbb{R}\}$

Exp 7. Non square system.

$$\begin{cases} 2x - 5y + 3z = 1 \\ x - 2y - 2z = 8 \end{cases}$$

A non square system $\Leftrightarrow$ nbr(eq^n) < nbr(var)

can be inconsistent or dependent

But never independent.

Exp 8. (2 free variables)

$$\begin{cases} x - 2y - 3z = 2 \\ 2x - 4y - 6z = 4 \end{cases}$$

$$\begin{bmatrix} 1 & -2 & -3 & 2 \\ 2 & -4 & -6 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x - 2y - 3z = 2 \rightarrow$ Dependent

$y = c$, $z = d \Rightarrow x = 2c + 3d + 2$

$SS = \{(2c + 3d + 2, c, d) : c, d \in \mathbb{R}\}$