9.8 Inverse of a Matrix.

Objectives

Define the inverse of a matrix, its properties & its applications

Identity matrix, properties, Inverse of a matrix, Its properties, how to find the inverse, Its applications

Identity Matrix.

$$I_n = \underbrace{\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}}_{n \text{ col}}$$ is the identity matrix of order n.

Properties of I_n.

1) A $n\times n$-matrix, $\Rightarrow$ $AI_n = I_n A = A$.

2) A is $m\times n$, B is $n\times p$ $\Rightarrow$ $AI_n = A$, $I_n B = B$

Inverse of a Matrix.

Given an $n\times n$-matrix A, then an $n\times n$-matrix B is the inverse of A if $A.B = I_n$ & $B.A = I_n$

In this case, B is written A^{-1}.

Note. To have an inverse, a matrix needs to be square.

Exp 1. Let A be an $n \times n$-matrix such that $A^3 = I_n$. Show that A has an inverse.

$$A^3 = I_n$$

$$\left.\begin{array}{l} A \cdot A^2 = I_n \\ A^2 \cdot A = I_n \end{array}\right| \Rightarrow \boxed{A^{-1} = A^2}$$

Note. Not all square matrices have inverses.

A is singular or non regular $\iff$ A has no inverse.

A is non-singular or regular $\iff$ A has an inverse.

How to find the inverse of a matrix?

2 Methods
- → A method for a 2×2-matrix
- → A general method for any $n\times n$-matrix

Inverse of a 2×2-matrix.

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is non singular if $\delta(A) = ad - bc \neq 0$

In this case $\boxed{A^{-1} = \frac{1}{\delta(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$

Ex 2. $A = \begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix} \Rightarrow \delta(A) = 4 - (-6) = 10 \Rightarrow A^{-1} = \frac{1}{10} \begin{pmatrix} 4 & -2 \\ 3 & 1 \end{pmatrix}$

$$= \begin{pmatrix} \frac{4}{10} & \frac{-2}{10} \\ \frac{3}{10} & \frac{1}{10} \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{3}{10} & \frac{1}{10} \end{pmatrix}$$

Gauss-Jordan Method of finding the inverse. (General)

1) Put I_n next to A, $[A \,\vdots\, I_n]$.

2) Use elementary row operations to change A to I_n, to get $[I_n \,\vdots\, B]$

3) $B = A^{-1}$

4) If we cannot get $I_n \Rightarrow A$ has no inverse.

Exp 3. Find the inverse of $A = \begin{pmatrix} 2 & 7 \\ 1 & 4 \end{pmatrix}$.

$$\left[\begin{array}{cc|cc} 2 & 7 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right]. \quad \text{We want to get } \left[\begin{array}{cc|cc} 1 & 0 & \times & \times \\ 0 & 1 & \times & \times \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 2 & 7 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|cc} 1 & 4 & 0 & 1 \\ 2 & 7 & 1 & 0 \end{array}\right] \xrightarrow{-2R_1 + R_2}$$

$$\left[\begin{array}{cc|cc} 1 & 4 & 0 & 1 \\ 0 & -1 & 1 & -2 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{cc|cc} 1 & 4 & 0 & 1 \\ 0 & 1 & -1 & 2 \end{array}\right] \xrightarrow{-4R_2 + R_1}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 4 & -7 \\ 0 & 1 & -1 & 2 \end{array}\right] \Rightarrow A^{-1} = \begin{pmatrix} 4 & -7 \\ -1 & 2 \end{pmatrix}$$

A^{-1}

Exp 4 Using the Gauss-Jordan Method find the inverse of

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -2 & -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[-2R_1+R_3]{-2R_1+R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -2 & 1 & 0 \\ 0 & 0 & 3 & -3 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3/3]{R_2/-2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3/2 & 1 & -1/2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1/3 \end{array}\right) \xrightarrow[-3/2 R_3 + R_2]{-R_3 + R_1}$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & -1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1 & 0 & -1/3 \end{array}\right)$$ - under form $[I_3 | B]$

$$\Rightarrow A^{-1} = \begin{pmatrix} 0 & 0 & 1/3 \\ -1/2 & -1/2 & 1/2 \\ 1 & 0 & -1/2 \end{pmatrix}$$

Solving Linear Systems Using Inverses.

$$(S) \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = c_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = c_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = c_n \end{cases}$$

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{pmatrix}, \quad X=\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad C=\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

Coefficient Matrix — Variable Matrix — Constant Matrix

The system (s) can be written in matrix form as

$$\boxed{AX=C}$$

<u>Result</u>. If A is non singular $\Rightarrow$ $\boxed{X=A^{-1}.C}$

<u>Because</u>. $AX=C \Rightarrow A^{-1}(AX)=A^{-1}.C$

$$I_n X = A^{-1}C$$

$$\boxed{X=A^{-1}C}$$

<u>Exp 5</u>. Solve $\begin{cases} x + z = 2 \\ 2x - 2y - z = 3 \\ 3z = -6 \end{cases}$ using inverses

$A=\begin{pmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{pmatrix}$, We saw in exp 4 that $A^{-1}=\begin{pmatrix} 0 & 0 & 1/3 \\ -1/2 & -1/2 & 1/2 \\ 1 & 0 & -1/2 \end{pmatrix}$

<u>Then</u>

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = X = A^{-1}.C = \begin{pmatrix} 0 & 0 & 1/3 \\ -1/2 & -1/2 & 1/2 \\ 1 & 0 & -1/2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix} = \begin{pmatrix} -2 \\ -1-3/2-3 \\ 2+3 \end{pmatrix} = \begin{pmatrix} -2 \\ -11/2 \\ 5 \end{pmatrix}$$

Properties of Inverse & consequences.

1) $AA^{-1} = I_n$ & $A^{-1} \cdot A = I_n$

2) $(A^{-1})^{-1} = A$

3) $(AB)^{-1} = B^{-1}A^{-1}$

4) $I_n^{-1} = I_n$

5) $(A+B)^{-1} \neq A^{-1} + B^{-1}$

6) $AB = C$ & B nonsingular $\Rightarrow A = CB^{-1}$

7) $AB = 0 \not\Rightarrow A = 0$ or $B = 0$

8) $AB = 0$ & A or B nonsingular $\rightarrow A = 0$ or $B = 0$

9) A has no inverse if a) one row is all zeros
or b) 2 rows are equal
or c) One row is a multiple of another row.

10) $$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}^{-1} = \begin{pmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{pmatrix}$$

<u>Exercise 1</u> $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & -1 \\ 3 & 6 & -n \end{pmatrix}$ & $A^{-1} = \begin{pmatrix} 0 & 2 & m \\ -1 & -1 & 1 \\ -3 & 0 & 1 \end{pmatrix}$

Find m & n.

Don't try to calculate the inverse of A, but use

$A \cdot A^{-1} = I_n$ &/or $A^{-1}A = I_n$

$$(C_{ij}) = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & -1 \\ 3 & 6 & -n \end{pmatrix} \cdot \begin{pmatrix} 0 & 2 & m \\ -1 & -1 & 1 \\ -3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \boxed{0} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$C_{13} = (1 \quad 2 \quad -1) \cdot \begin{pmatrix} m \\ 1 \\ 1 \end{pmatrix} = 0 \iff m + 2 - 1 = 0$$

$$\boxed{m = -1}$$

$$C_{31} = (3 \quad 6 \quad -n) \cdot \begin{pmatrix} 0 \\ -1 \\ -3 \end{pmatrix} = 0 \iff -6 + 3n = 0$$

$$\boxed{n = \frac{6}{3} = 2}$$

<u>Exercise 2.</u> $A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 4 \\ 0 & 4 & 3 \end{pmatrix}$ Find the sum of all elements in 1st row of A^{-1}.