— 9.5 Nonlinear systems of Equations —

A nonlinear system of equations is a system where one or more equations are nonlinear.

Objective:

To learn to solve some type of nonlinear systems of degree 2.

We solve these systems by substitution & elimination or a combination of both.

* If one equation is linear or of type $xy = a$ the substitution is used.

* If all terms contain only x^2 or y^2 $\Rightarrow$ the elimination is used.

The solution set is the set all intersection points of the graphs of the two equations.

Exp 1. Solve

$$\begin{cases} y = x^2 - x - 1 & (1) \\ 3x - y = 4 & (2) \end{cases}$$

Both methods can be used here.

Let use the substitution method

(1) is solved for y, substitute y in (2)

$$3x - (x^2 - x - 1) = 4$$

$$3x - x^2 + x + 1 = 4$$

$$x^2 - 4x + 3 = 0$$

$$(x - 3)(x - 1) = 0$$

$x = 3$, $x = 1$

Hence

$x = 3 \Rightarrow$

$y = 3^2 - (3) - 1 = 5$

$\Rightarrow (3, 5)$

$x = 1$

$y = 1^2 - 1 - 1 = -1$

$(1, -1)$

$SS = \{(3,5), (-1,1)\}$

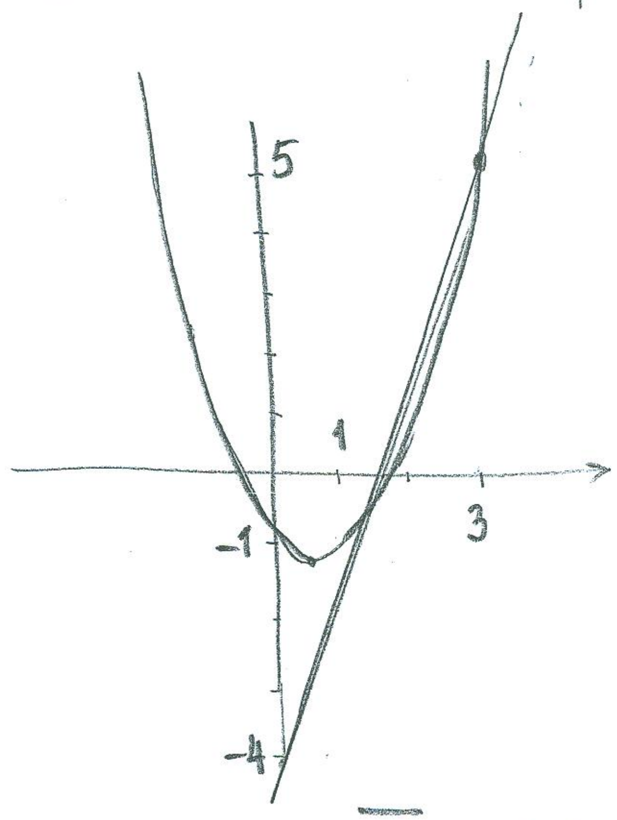
The geometric interpretation is as follows

$y = x^2 - x - 1$ is a parabola of vertex $(\frac{1}{2}, \frac{5}{4})$

$= (x - \frac{1}{2})^2 - \frac{5}{4}$

$3x - y = 4$ is a line whose slope is 3 & y-int -4

$y = 3x - 4$



Possible intersection of a parabola & a line

No sol^n

1 sol^n

2 sol^n

Exp 2. Solve

$$\begin{cases} 2x^2 + 3y^2 = 21 & (1) \\ x^2 + 2y^2 = 12 & (2) \end{cases}$$

The elimination method is better suited here

$$\begin{array}{lr} (1) & 2x^2 + 3y^2 = 21 \\ (2) \times -2 & -2x^2 - 4y^2 = -24 \\ \hline & -y^2 = -3 \end{array} \Rightarrow y = \pm\sqrt{3}$$

$y = -\sqrt{3}$

$\Rightarrow y^2 = 3$

$\xrightarrow{(2)}$ $x^2 + 6 = 12$

$x^2 = 6 \Rightarrow x = \pm\sqrt{6}$

$(-\sqrt{3}, -\sqrt{6})$, $(-\sqrt{3}, \sqrt{6})$

$y = \sqrt{3} \Rightarrow y^2 = 3$

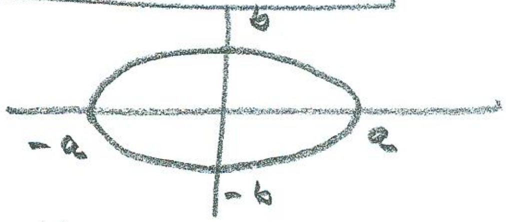
$\Rightarrow x^2 + 6 = 12$

$\Rightarrow x^2 = 6 \Rightarrow x = \pm\sqrt{6}$

$(\sqrt{3}, -\sqrt{6})$, $(\sqrt{3}, \sqrt{6})$

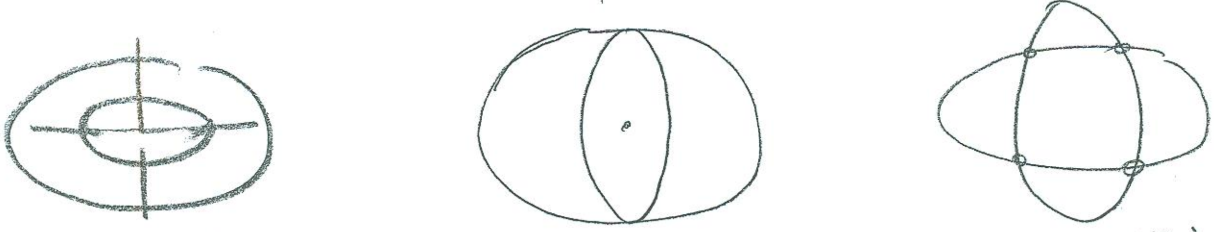
To visualize the graphs we need to know that $\boxed{mx^2 + ny^2 = k}$, where $m, n, k > 0$, is an <u>ellipse</u> of <u>center (0,0)</u>

Its standard equation is $\boxed{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1}$



So this system has ellipses that intersept is 4 <u>points</u>.

Possible intersections of 2 ellipses with center (0,0)



No sol   2 sol.   4 solⁿ

Exp 3 $x^2+y^2=4$

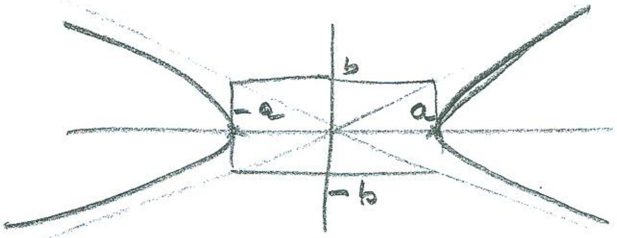
$2x^2-y^2=8$

Two solutions $(x,y)=(-2,0)$, $(2,0)$

To visualize, we need to remember that

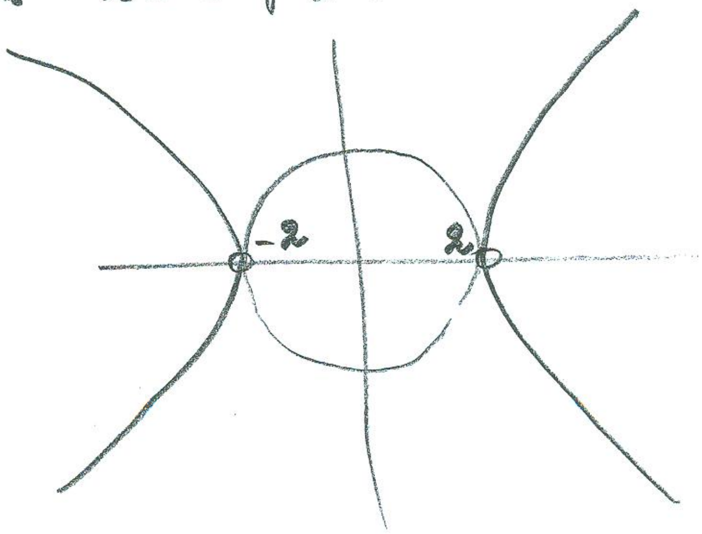
$mx^2-ny^2=k$, $m,n,k>0$ is a hyperbola

Its standard eqⁿ is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$



The system consists a circle $x^2+y^2=4$

& a hyperbola $2x^2-y^2=8$



Exp 4 Solve

$$\begin{cases} 4x^2 + 9y^2 = 36 \\ x^2 - y^2 = 25 \end{cases}$$

Ans. No sol^{ns}

Exp 5 Equation with xy.

$$\begin{cases} x^2 + xy + y^2 = 21 & (1) \\ x^2 - xy + y^2 = 9 & (2) \end{cases}$$

Notice that we can eliminate x^2 & y^2 at once.

$$\begin{array}{lr} (1) & x^2 + xy + y^2 = 21 \\ (2)\times - & -x^2 + xy - y^2 = -9 \\ \hline & 2xy \quad = 12 \end{array} \Rightarrow xy = 6 \Rightarrow y = 6/x \quad (3)$$

Substitute in (2)

$$x^2 - 6 + \frac{36}{x^2} = 9 \iff x^4 - 6x^2 + 36 = 9x^2$$

$$x^4 - 15x^2 + 36 = 0 \iff (x^2 - 12)(x^2 - 3) = 0$$

$x^2 = 12$

$x = \pm 2\sqrt{3}$

$x^2 = 3$

$x = \pm\sqrt{3}$

$x = -2\sqrt{3}$

$\overset{3}{\Rightarrow} y = \frac{6}{-2\sqrt{3}} = -\frac{3\sqrt{3}}{3} = -\sqrt{3}$

$\boxed{(-2\sqrt{3}, -\sqrt{3})}$

$x = 2\sqrt{3}$

$y = \sqrt{3}$

$\boxed{(2\sqrt{3}, \sqrt{3})}$

$x = -\sqrt{3}$

$y = -\frac{6}{\sqrt{3}} = -2\sqrt{3}$

$\boxed{(-\sqrt{3}, -2\sqrt{3})}$

$x = \sqrt{3}$

$y = 2\sqrt{3}$

$\boxed{(\sqrt{3}, 2\sqrt{3})}$

All the previous systems were solved in the set of real numbers, we solve the following system in the set of complex numbers $\mathbb{C}$.

Exp 7 $\begin{cases} x^2 + y^2 = 6 \\ 3x^2 + 2y^2 = 8 \end{cases}$

(1) $\times -2$ $\quad -2x^2 - 2y^2 = -12$

(2) $\quad 3x^2 - 2y^2 = 8$

$x^2 = -4 \Rightarrow x = \pm 2i$

$x = -2i$

(1) $\Rightarrow (-2i)^2 + y^2 = 6$

$-4 + y^2 = 6$

$y^2 = 10$

$y = \pm\sqrt{10}$

$(-2i, -\sqrt{10}), (-2i, \sqrt{10})$

$x = 2i$

$x^2 = -4$

$y = \pm\sqrt{10}$

$(2i, -\sqrt{10}), (2i, \sqrt{10})$

Applications

Exp Find 2 nbrs whose sum is 10, whose square differ by 28.

Ans $\begin{cases} x + y = 10 \\ x^2 - y^2 = 28 \end{cases}$ solve . . .

Exp 6. Abs. Val.

$$\begin{cases} x^2 + y^2 = 16 & (1) \\ |x| + y = 4 & (2) \end{cases}$$

Solve (2) for $|x|$.

$|x| = 4 - y$

$x^2 = (|x|)^2 = (4-y)^2 = 16 - 8y + y^2$, subst in (1)

$16 - 8y + y^2 + y^2 = 16$

$y^2 - 4y = 0$

$y(y-4) = 0$

$y = 0$

$|x| = 4 - 0$

$\Rightarrow x = \pm 4$

$(-4, 0), (4, 0)$

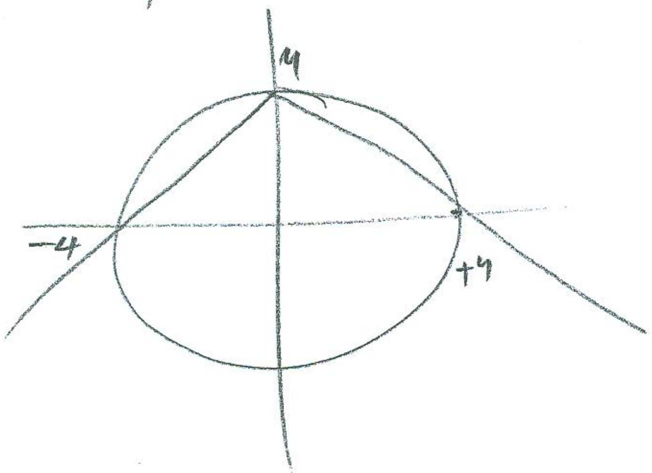
$y = 4$

$|x| = 4 - 4$

$|x| = 0$

$x = 0$

$(0, 4)$



Exp For what value of b, the line $x+2y=b$ touch the circle $x^2+y^2=9$ in only one pt., in 2 pts, at no point

Exp Solve $\begin{cases}(x+3)^2+(y-4)^2=20\\(x+4)^2+(y-3)^2=26\end{cases}$

Ans $(-5,8),(1,2)$

Exp $\begin{cases}x^2-3xy+y^2=5\\x^2-xy-2y^2=0\end{cases}$ Ans $(-1,1),(1,-1)$

(Factor)

Exp. $\begin{cases}3x^2+2xy-5y^2=11\\x^2+3xy+y^2=11\end{cases}$ Ans $(2,1),(-2,1)$

(Subtract the eqns)