— 9.3 Determinants —

The determinants are numbers associated with square matrices.

Objectives.
Define the determinant of a square matrice. Its properties.
How to compute it.

Determinant of a 2x2-matrix.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad , \quad |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Exp1. If $A = \begin{pmatrix} x & 3 \\ -1 & 5 \end{pmatrix}$ & $|A| = 33$. Find x.

$$33 = |A| = 5x + 3 \Rightarrow 5x = 30 \Rightarrow \boxed{x = 6}$$

Determinant of a 3×3-matrix.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\begin{array}{ccccc} + & + & + & - & - \quad - \\ \end{array}$$

$$|A| = \left|\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}\right. \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31}$$

Exp2.

$$A = \left(\begin{matrix} 2 & -1 & 3 \\ -2 & 4 & 5 \\ 6 & 3 & 1 \end{matrix}\right) \begin{matrix} 2 & -1 \\ -2 & 4 \\ 6 & 3 \end{matrix}$$

$$|A| = 2 \cdot 4 \cdot 1 + (-1) \cdot 5 \cdot 6 + 3 \cdot (-2) \cdot 3 - 1 \cdot (-2) \cdot (-1) - 2 \cdot 5 \cdot 3 - 3 \cdot 4 \cdot 6$$

$$= 8 - 30 - 18 - 2 - 30 - 72 = -72 - 72 = \boxed{-144}$$

Minors. Let A be an $n\times n$-matrix.
The minor M_{ij} of a_{ij} in A is the determinant of a the matrix remaining from A by erasing the i^{th} row & the j^{th} column.

Cofactors. The cofactor C_{ij} of a_{ij} of A is

$$C_{ij} = (-1)^{i+j} M_{ij}$$

$$C_{ij} = \begin{cases} M_{ij} & \text{if } i+j:\text{ even} \\ -M_{ij} & \text{if } i+j:\text{ odd} \end{cases}$$

Exp 3.

$$A = \begin{pmatrix} 2 & -1 & 5 \\ 4 & 3 & -7 \\ 8 & -7 & 6 \end{pmatrix}$$

Find M_{23}, M_{33}, C_{23}, C_{33}

$$M_{23} = \begin{vmatrix} 2 & -1 & \cancel{5} \\ \cancel{4} & \cancel{3} & \cancel{-7} \\ 8 & -7 & \cancel{6} \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 8 & -7 \end{vmatrix} = -14 + 8 = -6$$

$$C_{23} = (-1)^{2+3} M_{23} = (-1)(-6) = \boxed{6}$$

$$M_{33} = \begin{vmatrix} 2 & -1 & \cancel{5} \\ 4 & 3 & \cancel{-7} \\ \cancel{8} & \cancel{-7} & \cancel{6} \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 4 & 3 \end{vmatrix} = 6 - (-4) = 10$$

$$C_{33} = (-1)^{3+3} M_{33} = (+1)\, 10 = \boxed{10}$$

Evaluating the determinant of an nxn-matrix:

The determinant can be calculated following any row or any column as follows.

$$|A| = a_{r1} C_{r1} + a_{r2} C_{r2} + \dots + a_{rn} C_{rn}$$

(following r th Column)

or

$$|A| = a_{1s} C_{1s} + a_{2s} C_{2s} + \dots + a_{ns} C_{ns}.$$

Exp. Compute the determinant of $A = \begin{pmatrix} 2 & 3 & -1 \\ 4 & -2 & 3 \\ 2 & 4 & 0 \end{pmatrix}$

We choose the third row (or the third column).

$$|A| = 2 C_{31} + 4 C_{32} + 0 C_{33} = 2(-1)^{3+1} M_{31} + 4(-1)^{3+2} M_{32}$$

$$= 2 \begin{vmatrix} 3 & -1 \\ -2 & 3 \end{vmatrix} - 4 \begin{vmatrix} 2 & -1 \\ 4 & 3 \end{vmatrix} = 2(9-2) - 4(6+4)$$

$$= 14 - 40 = \boxed{-26}$$

Row & Column Operations. A an nxn matrix

1) If B is obtained by interchanging 2 rows or 2 columns of A then $|B| = -|A|$

Exp. $\begin{vmatrix} c & d \\ a & b \end{vmatrix} \xrightarrow{R_1 \to R_2} \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

2) If B is formed by multiplying 1 row or column of A by k, then $|B| = k|A|$

Eg.
$$\begin{vmatrix} a & b \\ kc & kd \end{vmatrix} = k\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

In other words, we can factor out a common factor from any row or any column.

3) If B is obtained by adding a multiple of a row to another row (or column) of A, then
$$|A| = |B|$$

Exp
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c+ka & d+ka \end{vmatrix}$$

4) If A is $n \times n$ $\Rightarrow$
$$|kA| = k^n |A|$$

Exp
$$\begin{vmatrix} ka & kb \\ kc & kd \end{vmatrix} = k^2\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

5) Determinant of a triangular matrix.

$$\begin{vmatrix} a_{11} & a_{12} & \cdot & a_{1n} \\ 0 & a_{22} & & \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & a_{nn} \end{vmatrix} = a_{11} a_{22} \cdots a_{nn}$$

(Product of all diagonal elements)

How to evaluate a determinant

Directly
Choose a row or col. with the most zeros & compute $|A|$

Introducing more zeros & then compute $|A|$

Put in **triangular form** & then compute $|A|$

Exp **Introducing zeros.**

Compute the derminant of $A = \begin{pmatrix} 3 & 2 & -2 \\ -1 & -1 & 4 \\ 2 & 4 & -1 \end{pmatrix}$

Let introduce 2 zeros in col 2 of A.

$$\begin{vmatrix} 3 & 2 & -2 \\ -1 & -1 & 4 \\ 2 & 4 & -1 \end{vmatrix} \underset{4R_2+R_3}{\overset{2R_2+R_1}{=}} \begin{vmatrix} 1 & 0 & 6 \\ -1 & -1 & 4 \\ -2 & 0 & 15 \end{vmatrix} = (-1)\, C_{22}$$

$$= (-1)(-1)^{2+2} M_{22} = -\begin{vmatrix} 1 & 6 \\ -2 & 15 \end{vmatrix} = -(15+12) = \boxed{-27}$$

We can choose to introduce zeros in row 2

$$\begin{vmatrix} 3 & 2 & -2 \\ -1 & -1 & 4 \\ 2 & 4 & -1 \end{vmatrix} \underset{4C_1+C_3}{\overset{-C_1+C_2}{=}} \begin{vmatrix} 3 & -1 & 10 \\ -1 & 0 & 0 \\ 2 & 2 & 7 \end{vmatrix} = (-1)(-1)^{1+2} M_{21}$$

$$= (-1)(-1) \begin{vmatrix} -1 & 10 \\ 2 & 7 \end{vmatrix} = -27$$

Following the 2nd row

Properties.

1) $|A| = 0$ if

a) one row or column is all zeros.

b) 2 rows (or 2 columns) are equal.

c) 2 rows (or 2 columns) are multiple of each other.

2) A nonsingular $\Longleftrightarrow$ $|A| \neq 0$

3) $|I_n| = 1$

4) $|A^{-1}| = \dfrac{1}{|A|}$

5) $|AB| = |A| \cdot |B|$

Exercise.

A, B 4×4-matrices & $|A| = 4$, $|B| = 5$.

Evaluate $|AB| - |2A^{-1}|$

Solⁿ. $|AB| = |A| \cdot |B| = 4 \cdot 5 = 20$

$$|2A^{-1}| = 2^4 \, |A^{-1}| = 16 \cdot \frac{1}{|A|} = 16 \cdot \frac{1}{4} = 4$$

$$|AB| - |2A^{-1}| = 20 - 4 = \boxed{16}$$

Exercise Find the cofactor of 4 in $\begin{pmatrix} 3 & -6 & 5 & -1 \\ 0 & 2 & -1 & 3 \\ -6 & 4 & 2 & 0 \\ -7 & 3 & 1 & 1 \end{pmatrix}$

Solⁿ. $$C_{32} = (-1)^{3+2} M_{32} = - \begin{vmatrix} 3 & 5 & -1 \\ 0 & -1 & 3 \\ -7 & 1 & 1 \end{vmatrix}$$

follow

$$\overset{\downarrow}{\underset{\text{2nd row}}{=}} -\left(0\,C_{21} + (-1)\,C_{22} + 3\,C_{23}\right) = -\left(-M_{22} - 3\,M_{23}\right)$$

$$= M_{22} + 3\,M_{23} = \begin{vmatrix} 3 & -1 \\ -7 & 1 \end{vmatrix} + 3\begin{vmatrix} 3 & 5 \\ -7 & 1 \end{vmatrix} = (3-7) + 3(3+35)$$

$$= -4 + 3(38) = -4 + 114 = \boxed{110}$$

__Exercise.__ Solve $\begin{vmatrix} 1 & -1 & 2 \\ 0 & x & 1 \\ 3 & 2 & x-1 \end{vmatrix} = -17$

Sol^{n}. Expand following 1st Column,

$$1.C_{11} + 0\,C_{21} + 3\,C_{31} = M_{11} + 3\,M_{31} = \begin{vmatrix} x & 1 \\ 2 & x-1 \end{vmatrix} + 3\begin{vmatrix} -1 & 2 \\ x & 1 \end{vmatrix}$$

$$= x(x-1) - 2 + 3(-1-2x) = x^2 - 7x - 5 = -17$$

$$\Longleftrightarrow \quad x^2 - 7x + 12 = 0$$

$$(x-3)(x-4) = 0$$

$$\boxed{x = 3 \quad , x = 4}$$