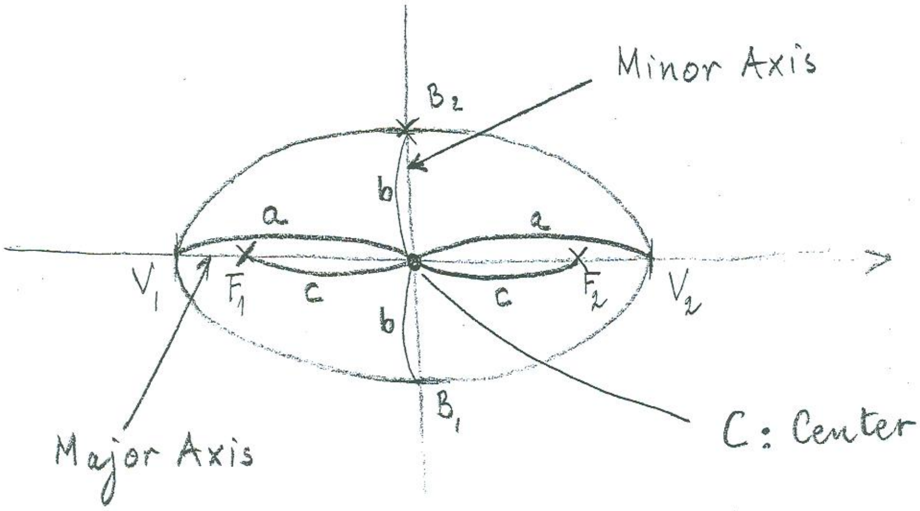
— 10.2. Ellipses —

Objective. Geometric Definition, Equations

Geometric Definition.

Given 2 points F_1, F_2, the ellipse of foci F_1, F_2 and distance $2a$ is the set of points P whose sum of distances to F_1 and F_2 is $2a$.

$$\mathcal{E}(F_1, F_2; 2a) = \{ P : d(P, F_1) + d(P, F_2) = 2a \}$$



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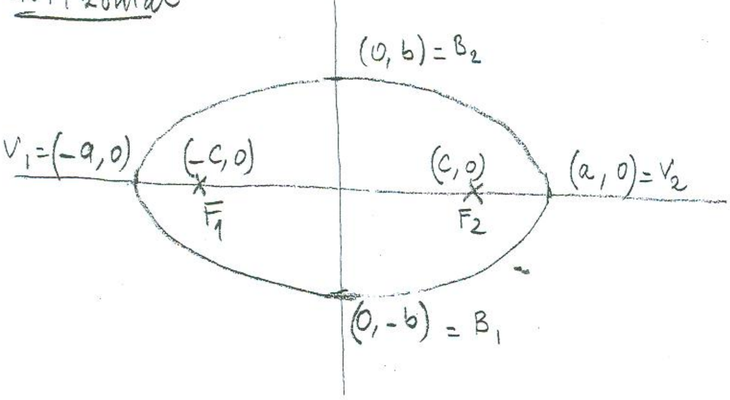
Ellipse of Center (0,0).

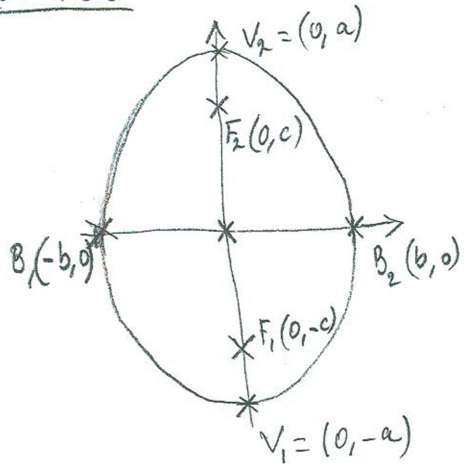
horizontal



$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

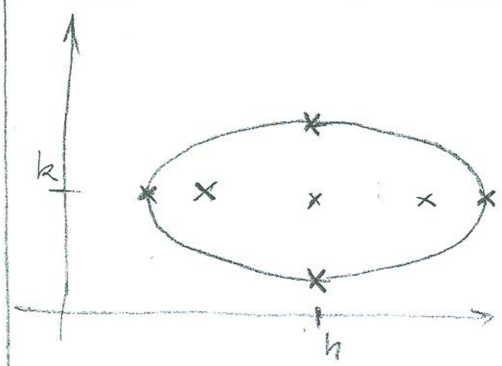
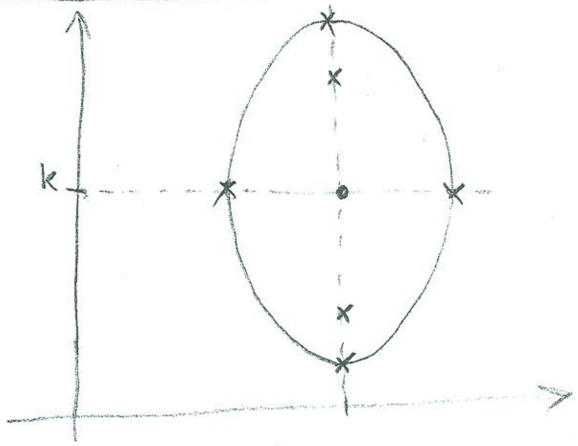
$$\boxed{b^2 + c^2 = a^2}$$

Vertical



$$\boxed{\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1}$$

Ellipses of Center (h, k).

Major Axis	Horizontal	Vertical
Graph		
Equation	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$
Vertices	$V = (h \pm a, k)$	$V = (h, k + a)$
Foci	$F = (h \pm c, k)$ $c = \sqrt{a^2 - b^2}$	$F = (h, k \pm c)$ $c = \sqrt{a^2 - b^2}$
length of Major axis	$2a$	$2a$
Length of Minor axis	$2b$	$2b$

Eccentricity.

This is a parameter which measures its closeness to being a circle.

$$\boxed{e = \frac{c}{a}}$$

& satisfies 1) $0 < e < 1$

2) $e \simeq 0 \Rightarrow$ ellipse $\simeq$ circle

3) $e \simeq 1 \Rightarrow$ ellipse very flat.

Properties.

1) $\dfrac{(x-h)^2}{K} + \dfrac{(y-h)^2}{L} = 1$ & $K, L > 0$

$\Longrightarrow \quad a^2 = \max\{K, L\}$, $b^2 = \min\{K, L\}$

2) $Ax^2 + Bx + Cy^2 + Dy + E = 0$

- gives an ellipse if A & B have same sign

& equivalent to

$$A(x-h)^2 + B(y-k)^2 = F > 0$$

If $F = 0 \Rightarrow$ Graph $= \{(h, k)\}$ 1 point

If $F < 0 \Rightarrow$ No graph.

3) If $\underbrace{F_1(h, y_1)\ ,\ F_2(h, y_2)}_{\text{same } x\text{-coordinates}}$ or $\underbrace{V_1(h, k-a)\ ,\ V_2(h, k+a)}$ $\Bigg\}$ a^2 under $(y-k)^2$

4) If $F(x_1, \underbrace{k)\ ,\ F(x_2, k}$) or $V_1(x_1', k)\ ,\ V_2(x_2', k)$ $\Bigg\} \Rightarrow$ a^2 under $(x-h)^2$

5) Center $= C =$ midpoint (F_1, F_2) $=$ midpoint (V_1, V_2)
$=$ midpoint (B_1, B_2).

6) Major Axis $= 2a$
Minor Axis $= 2b$

Exercise 1 Find the center, vertices, foci of the ellipse

$$4x^2 + 9y^2 - 8x + 36y + 4 = 0$$

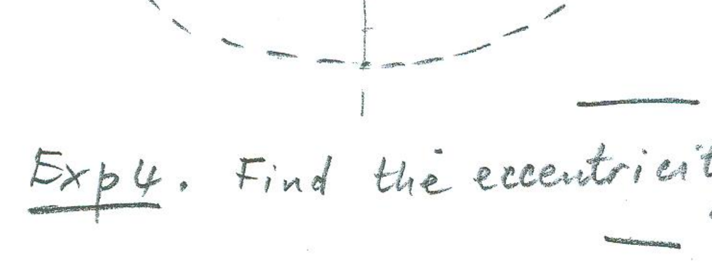
Exp2. An ellipse with foci $F_1(0,4)$, $F_2(6,4)$ & the minor axis has length 8.
Find its eq^n.

Exp3. Give the domain & the range of $\frac{y}{4} = \sqrt{1 - \frac{x^2}{25}}$

$\Leftrightarrow y \geq 0 \quad \frac{y^2}{16} = 1 - \frac{x^2}{25} \quad \Leftrightarrow \quad \frac{x^2}{25} + \frac{y^2}{16} = 1 \quad \& \; y \geq 0$

$\Rightarrow a = 5, \quad b = 4$

horizontal Ellipse.



Dom = $[-5, 5]$

Range : $[0, 4]$

Exp4. Find the eccentricity of $3x^2 + 10y^2 = 50$

Exp 5. An ellipse of foci $F_1 = (-3,-3)$ & $F_2\ (7,-3)$ passes through $(2,-7)$. Find its equation & its center & its vertices.

$$C = \left(\frac{-3+7}{2}, \frac{-3+(-3)}{2}\right) = (2,-3)$$

$F_1 = (-3,-3)$, $F_2 = (7,-3)$ $\Rightarrow$ a^2 under x^2 $\Rightarrow$ $\boxed{\dfrac{(x-2)^2}{a^2} + \dfrac{(y+3)^2}{b^2} = 1}$

$$c = \text{dist}(F_1, C) = \sqrt{(2-(-3))^2 + (-3+3)^2}$$
$$= \sqrt{25+0} = \sqrt{25} = \boxed{5}$$

$$\boxed{b^2 = a^2 - c^2 = a^2 - 25}$$

$(2,-7) \in$ Ellipse

$$\frac{(2-2)^2}{a^2} + \frac{(-7+3)^2}{a^2-25} = 1$$

$\Rightarrow a^2 - 25 = 4^2 = 16 \Rightarrow a^2 = 41$

$\Rightarrow b^2 = a^2 - 25 = 16$

$$\boxed{\frac{(x-2)^2}{41} + \frac{(y+3)^2}{16} = 1}$$

Exp 6. Find the eq^n of the ellipse of vertices $(-4,0)$, $(4,0)$ & eccentricity $e = 1/2$.

Center $(0,0)$,

Ellipse is horizontal. &

$$\boxed{a = 4}$$

$$\frac{1}{2} = e = \frac{c}{a} = \frac{c}{4} \Rightarrow \boxed{c = 2}$$

$\Rightarrow b^2 = a^2 - c^2$
$= 16 - 4 = 12$

$$\boxed{\frac{x^2}{16} + \frac{y^2}{12} = 1}$$