PERTURBATIVE TREATMENT OF COULOMB TYPE POTENTIAL IN THE PRESENCE OF A BACKGROUND HARMONIC OSCILLATOR FIELD

Harry A. Mavromatis* and Mohammad A. Al-Solami

Physics Department
King Fahd University of Petroleum & Minerals
Dhahran, Saudi Arabia

الخلاصـــة :

يمكن تمثيل الجهد الحاوي للكواركات quarks بجهد الهزاز التوافقي . ويمكن تشبيه جهد أقصر مجال للتفاعل بين الكواركات بجهد كولومب . وقد تعاملنا مع هذا الجهد الأخير باستعمال نظرية الاضطرابات الصغيرة وحصلنا أثناء هذه الدراسة على علاقات بين فواصل الطاقة المضطربة لأي عزم دائري (l) . وقد حسبنا مقدار الاضطراب المحدث بإدخال جهد كولومب وبإدخال جهد خطي وقد وضحنا أثر هـذيْن الاضطرابين بيانياً .

ABSTRACT

The confining potential for quarks may be simulated by a harmonic oscillator potential. The shorter range interaction between the quarks can in turn be simulated by a Coulomb potential. This latter potential we treat using perturbation theory, and in the process obtain some relations between perturbed energy spacings for general l. Finally we also numerically evaluate the perturbations introduced by a Coulomb and by a linear potential and illustrate these effects graphically.

*Address for correspondence:
KFUPM Box No. 58
King Fahd University of Petroleum & Minerals
Dhahran 31261
Saudi Arabia

H. A. Mavromatis and M. A. Al-Solami

PERTURBATIVE TREATMENT OF COULOMB TYPE POTENTIAL IN THE PRESENCE OF A BACKGROUND HARMONIC OSCILLATOR FIELD

INTRODUCTION

Quarks in nuclei or mesons satisfy two boundary conditions. On the one hand, at large distances, they are subject to a strong attractive force which keeps them together. This can be simulated by a harmonic oscillator field. On the other hand, at short distances, they experience a short-range attraction. This may be approximated by a Coulomb potential. These facts, and a recent preprint [1] in which this subject was studied motivated us to investigate this boundary-value problem using perturbation theory and drawing on a general result one of us derived [2] namely:

$$\int_0^\infty \exp(-x)x^\beta L_n^a(x)L_{n'}^{a'}(x)\,dx = \frac{\Gamma(n+a+1)^2\,\Gamma(n'+a'+1)}{n!\,n'!} \times \frac{\Gamma(\beta+1)\,\Gamma(n'-\beta+a')}{\Gamma(-\beta+a')\Gamma(a+1)} \times {}_3F_2(-n,\beta+1,\beta-a'+1;a+1,\beta-a'+1-n';1). \quad (1)$$

The ${}_3F_2$ in Equation (1) is a generalized hypergeometric [3] function which may be expressed in terms of a finite series since n in Equation (1) is an integer and generally:

$$ {}_pF_q(a_1\dots a_p;b_1\dots b_q;x) = 1 + \frac{\Gamma(a_1+1)\dots\Gamma(a_p+1)\Gamma(b_1)\dots\Gamma(b_q)}{\Gamma(a_1)\dots\Gamma(a_p)\Gamma(b_1+1)\dots\Gamma(b_q+1)}\frac{x}{1!} + \frac{\Gamma(a_1+2)\dots\Gamma(a_p+2)\Gamma(b_1)\dots\Gamma(b_q)}{\Gamma(a_1)\dots\Gamma(a_p)\Gamma(b_1+2)\dots\Gamma(b_q+2)}\frac{x^2}{2!} + \dots ,$$

Seven special cases where the ${}_3F_2$ in Equation (1) simplifies were given in reference 2. In the Appendix we list two additional cases where Equation (1) simplifies. These new results were obtained with the help of contiguity relations [4].

DISCUSSION AND RESULTS

Two quarks of mass m attracted to each other by a three-dimensional oscillator force $-k\vec{r}$ are mathematically equivalent in their relative motion to a particle of mass $\mu = m/2$, subject to the potential $\frac{1}{2}kr^2$, where $\vec{r} = \vec{r}_1 - \vec{r}_2$. We choose to work in units where $\mu = k = \hbar = 1$.

The radial eigenfunctions $R_{nl}(r)$ of the three-dimensional harmonic oscillator Schrödinger equation:

$$\left(\frac{\hat{p}^2}{2} + \frac{r^2}{2}\right) Y^l_{m_l}(\theta,\phi)R_{nl}(r) = \varepsilon_{nl}Y^l_{m_l}(\theta,\phi)R_{nl}(r) = (2n+l+\tfrac{3}{2})Y^l_{m_l}(\theta,\phi)R_{nl}(r),$$

$(n, l = 0, 1, \dots |m_l| \le l)$, are [5]

$$|nl\rangle = R_{nl}(r) = \left[\frac{2n!}{\Gamma(n+l+\frac{3}{2})^3}\right]^{\frac{1}{2}} r^l \exp(-\tfrac{1}{2}r^2)\,L_n^{l+\frac{1}{2}}(r^2).$$

Hence (in the above units) the unperturbed (relative) energy levels of a two-quark system can be written:

$$\varepsilon_{nl} = 2n+l+\tfrac{3}{2}\ .$$

If a perturbation V is added to this system, the first-order energy correction is given by:

$$\Delta E_{n,l}(V) = \langle nl|V|nl\rangle\ .$$

Thus, if a linear perturbation $V(r) = \alpha r$ is applied,

$$\langle nl|\alpha r|nl\rangle = \frac{\alpha n!}{\Gamma^3(n+l+\frac{3}{2})}\int_0^\infty \exp(-u)u^{l+1}L_n^{l+\frac{1}{2}}(u)L_n^{l+\frac{1}{2}}(u)\,du.$$

Considering the special case $\beta = l+1$ in Equation (1) one obtains the energy correction due to such a linear perturbation:

$$\Delta E_{n,l}(\alpha r) = -\frac{\alpha(l+1)!\,\Gamma(n-\frac{1}{2})}{2n!\,\Gamma(l+\frac{3}{2})\sqrt{\pi}} \times {}_3F_2(-n,l+2,\tfrac{3}{2};l+\tfrac{3}{2},\tfrac{3}{2}-n;1). \quad (2)$$

A simple subtraction and rearrangement of terms further yields:

$$\Delta E_{n+1,l}(\alpha r) - \Delta E_{n,l}(\alpha r) = \frac{\alpha 3(l+1)!\,\Gamma(n-\frac{1}{2})}{4\sqrt{\pi}(n+1)!\,\Gamma(l+\frac{3}{2})} \times {}_3F_2(-(n+1),l+2,\tfrac{3}{2};l+\tfrac{3}{2},\tfrac{3}{2}-n;1). \quad (3)$$

For $l = 0$, Equations (2), (3) reduce to very simple expressions if one uses Gauss's theorem [6]:

$${}_2F_1(\alpha,\beta;\gamma;1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}\ , \quad (4)$$

(where $\beta > 0, \gamma - \alpha - \beta > 0$). (These two $l = 0$ results are also given in reference 1):

$$\Delta E_{n,0}(\alpha r) = -\frac{\alpha \Gamma(n-\frac{1}{2})}{\pi n!}\, {}_3F_2(-n, 2, \tfrac{3}{2}; \tfrac{3}{2}, \tfrac{3}{2}-n; 1)$$

$$= -\frac{\alpha \Gamma(n-\frac{1}{2})}{\pi n!}\, {}_2F_1(-n, 2; \tfrac{3}{2}-n; 1)$$

$$= \frac{\alpha 4\Gamma(n+\frac{3}{2})}{\pi n!}, \qquad (5)$$

$$\Delta E_{n+1,0}(\alpha r) - \Delta E_{n,0}(\alpha r)$$

$$= \frac{\alpha 3\Gamma(n-\frac{1}{2})}{2\pi(n+1)!}\, {}_2F_1(-(n+1), 2; \tfrac{3}{2}-n; 1)$$

$$= \frac{\alpha 2\Gamma(n+\frac{3}{2})}{\pi(n+1)!}. \qquad (6)$$

Similarly if the unperturbed system is subjected to a Coulomb-type potential $V(r) = \lambda/r$,

$$\left\langle nl \left| \frac{\lambda}{r} \right| nl \right\rangle$$

$$= \frac{\lambda n!}{\Gamma^3(n+l+\frac{3}{2})} \int_0^\infty \exp(-u) u^l L_n^{l+\frac{1}{2}}(u) L_n^{l+\frac{1}{2}}(u)\, du.$$

Considering the special case $\beta = l$ in Equation (1) one obtains the energy correction due to a Coulomb potential:

$$\Delta E_{n,l}\left(\frac{\lambda}{r}\right) = \frac{\lambda l! \Gamma(n+\frac{1}{2})}{\sqrt{\pi}\, n! \Gamma(l+\frac{3}{2})}$$

$$\times\, {}_3F_2(-n, l+1, \tfrac{1}{2}; l+\tfrac{3}{2}, \tfrac{1}{2}-n; 1). \qquad (7)$$

Substracting and combining terms in this case one has:

$$\Delta E_{n+1,l}\left(\frac{\lambda}{r}\right) - \Delta E_{n,l}\left(\frac{\lambda}{r}\right)$$

$$= -\frac{\lambda l! \Gamma(n+\frac{1}{2})}{2\sqrt{\pi}(n+1)! \Gamma(l+\frac{3}{2})}$$

$$\times\, {}_3F_2(-(n+1), l+1, \tfrac{1}{2}; l+\tfrac{3}{2}, \tfrac{1}{2}-n; 1). \qquad (8)$$

To our knowledge Equations (2), (3), (7), and (8) are new, as well as useful results even though the ${}_3F_2$'s in each of these expressions involve finite series. Thus, if for instance $n = 0$, substituting into Equation (8) one has:

$$\Delta E_{1,l}\left(\frac{\lambda}{r}\right) - \Delta E_{0,l}\left(\frac{\lambda}{r}\right)$$

$$= -\frac{\lambda l! \Gamma(\frac{1}{2})}{2\sqrt{\pi}\, \Gamma(l+\frac{3}{2})}\, {}_3F_2(-1, l+1, \tfrac{1}{2}; l+\tfrac{3}{2}, \tfrac{1}{2}; 1)$$

$$= -\frac{\lambda l!}{2\Gamma(l+\frac{3}{2})} \left\{ 1 + \frac{(-1)(l+1)(\frac{1}{2})}{(l+\frac{3}{2})(\frac{1}{2})} \right\}$$

$$= -\frac{\lambda l!}{4\Gamma(l+\frac{5}{2})}$$

etc.

In his preprint [1] Fayyazuddin obtains two interesting new relations between linear and Coulomb energy corrections. In our notation these relations are as follows:

$$(n+l+\tfrac{3}{2}) \Delta E_{n,l}(\alpha r) - (n+1) \Delta E_{n+1,l}(\alpha r) - l \Delta E_{n,l+1}(\alpha r) = 0, \qquad (9)$$

$$(n+l+\tfrac{3}{2}) \Delta E_{n,l}\left(\frac{\lambda}{r}\right) - (n+1) \Delta E_{n+1,l}\left(\frac{\lambda}{r}\right) - (l+1) \Delta E_{n,l+1}\left(\frac{\lambda}{r}\right) = 0. \qquad (10)$$

If one writes the corrections involved using Equations (2), (7), one can show these two results (*i.e.* Equations (9), (10)), imply the contiguity relation:

$$(n+a-b+1)\, {}_3F_2(-n, a, b; a-b+1, b-n; 1)$$

$$-(n-b+1)\, {}_3F_2(-\{n+1\}, a, b; a-b+1, b-\{n+1\}; 1)$$

$$-\frac{a(a-2b+1)}{a-b+1}\, {}_3F_2(-n, a+1, b; a-b+2, b-n; 1) = 0, \qquad (11)$$

where the parametric excess [7] $s = 1-b$. (For general ${}_3F_2(\alpha, \beta, \gamma; \delta, \varepsilon; 1)$, $s \equiv \delta + \varepsilon - \alpha - \beta - \gamma$, and for Saalschutzian series $s = 1$.) The three-term contiguity relation Equation (11) can be easily verified analytically for the simple cases $n = 0, 1, 2$. We also verified it numerically for arbitrary values of a and b and various large values of the integer n.

A more general way to verify Equation (11) is as follows [9].

Use the relationships [10–12]:

$${}_1F_0(\alpha; z) = \sum_{p=0}^{\infty} \frac{\Gamma(p+\alpha)}{\Gamma(\alpha)} \frac{z^p}{p!} = (1-z)^{-\alpha},$$

$$\frac{\Gamma(n-z)}{\Gamma(-z)} = (-1)^n \frac{\Gamma(z+1)}{\Gamma(z+1-n)} \quad (n \text{ integer}),$$

$$\frac{\Gamma(z-n)}{\Gamma(z)} = (-1)^n \frac{\Gamma(1-z)}{\Gamma(1-z+n)} \quad (n \text{ integer}).$$

Multiplying Equation (11) by $\dfrac{\Gamma(n+1-b)}{\Gamma(1-b)}\dfrac{z^n}{n!}$ and summing over n, one obtains for the first term:

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+1-b)\,z^n(n+a-b+1)}{\Gamma(1-b)\,n!}$$
$$\times\, {}_3F_2(-n,a,b;a-b+1,b-n;1)$$
$$= a\,{}_2F_1(a+1,b;a-b+1;z)(1-z)^{b-1} + (1-b)$$
$$\times\, {}_2F_1(a,b;a-b+1;z)(1-z)^{b-2},$$

while for the second term:

$$-\sum_{n=0}^{\infty} \frac{\Gamma(n+1-b)\,z^n(n-b+1)}{\Gamma(1-b)\,n!}$$
$$\times\, {}_3F_2(-\{n+1\},a,b;a-b+1,b-\{n+1\};1)$$
$$= -bz^{-1}\,{}_2F_1(a,b+1;a-b+1;z)(1-z)^{b-1}$$
$$-\,(1-b)z^{-1}\,{}_2F_1(a,b;a-b+1;z)$$
$$\times\,(1-z)^{b-2} + z^{-1}\,{}_2F_1(a,b;a-b+1;z)(1-z)^{b-1},$$

and the third term

$$-\frac{a(a-2b+1)}{a-b+1}\sum_{n=0}^{\infty}\frac{\Gamma(n+1-b)}{\Gamma(1-b)\,n!}$$
$$\times\, {}_3F_2(-n,a+1,b;a-b+2,b-n;1)$$
$$= -\frac{a(a-2b+1)}{a-b+1}$$
$$\times\, {}_2F_1(a+1,b;a-b+2;z)(1-z)^{b-1}.$$

Thus one obtains:

$$a\,{}_2F_1(a+1,b;a-b+1;z)(1-z)^{b-1} + (1-b)$$
$$\times\, {}_2F_1(a,b;a-b+1;z)(1-z)^{b-2}$$
$$-\,bz^{-1}\,{}_2F_1(a,b+1;a-b+1;z)(1-z)^{b-1}$$
$$-\,(1-b)z^{-1}\,{}_2F_1(a,b;a-b+1;z)(1-z)^{b-2} + z^{-1}$$
$$\times\, {}_2F_1(a,b;a-b+1;z)(1-z)^{b-1} - \frac{a(a-2b+1)}{a-b+1}$$
$$\times\, {}_2F_1(a+1,b;a-b+2;z)(1-z)^{b-1}$$

If one now factors $(1-z)^{b-2}$ out of this expression, one can readily verify that what remains is a power series in z, (in fact beginning from z^{-1}) all of whose coefficients are 0.

The results of plotting Equations (2) and (7) are illustrated in Figures 1–4. In Figure 1 the first-order linear corrections of Equation (2) are plotted for the case $\alpha = 1$ and in Figure 2 the actual energy levels to first-order are shown for this value of α. Analogously in Figure 3 the first-order Coulomb corrections of Equation (7) are plotted for the case $\lambda = 1$ and in Figure 4 the actual energy levels to first-order are shown for this value of λ. Different columns correspond to increasing values of l (starting from $l = 0$), and the vertical scale corresponds to increasing values of n (starting from $n = 0$).

One notes the first-order linear correction increases as n and/or l increases. This can be understood by noting that as n, l increase the probability distribution $r^2R^2_{nl}(r)$ moves towards larger r. Thus for $n = 0$,

$$L_0^{l+\frac{1}{2}}(r^2) = \Gamma(l+\tfrac{3}{2}),$$

$$r^2R^2_{0l}(r) = \frac{2}{\Gamma(l+\frac{3}{2})}\,r^{2l+2}\exp(-r^2),$$

and

$$r_{\max} = \sqrt{l+1}, \quad \frac{1}{r_{\max}} = \frac{1}{\sqrt{l+1}}.$$

Hence the expectation value of r *i.e.* $\Delta E_{nl}(\alpha r)$ increases and analogously $\Delta E_{nl}(\lambda/r)$, decreases with increasing n, l.

An indication of the flexibility of the approach in this paper may be obtained by considering the case when the attractive perturbation is a bit stronger or weaker than λ/r, *i.e.* $\lambda/r^{1\pm\varepsilon}$.

In this case:

$$\Delta E_{nl}\left(\frac{\lambda}{r^{1\pm\varepsilon}}\right) = \frac{2\lambda n!}{\Gamma(n+l+\frac{3}{2})^3}\int_0^{\infty} r^{2l-1\mp\varepsilon}$$
$$\exp(-r^2)\,L_n^{l+\frac{1}{2}}(r^2)L_n^{l+\frac{1}{2}}(r^2)\,r^2\,\mathrm{d}r$$
$$= \frac{\lambda n!}{\Gamma(n+l+\frac{3}{2})^3}\int_0^{\infty} u^{l\mp\varepsilon/2}$$
$$\exp(-u)\,L_n^{l+\frac{1}{2}}(u)\,L_n^{l+\frac{1}{2}}(u)\,\mathrm{d}u,$$

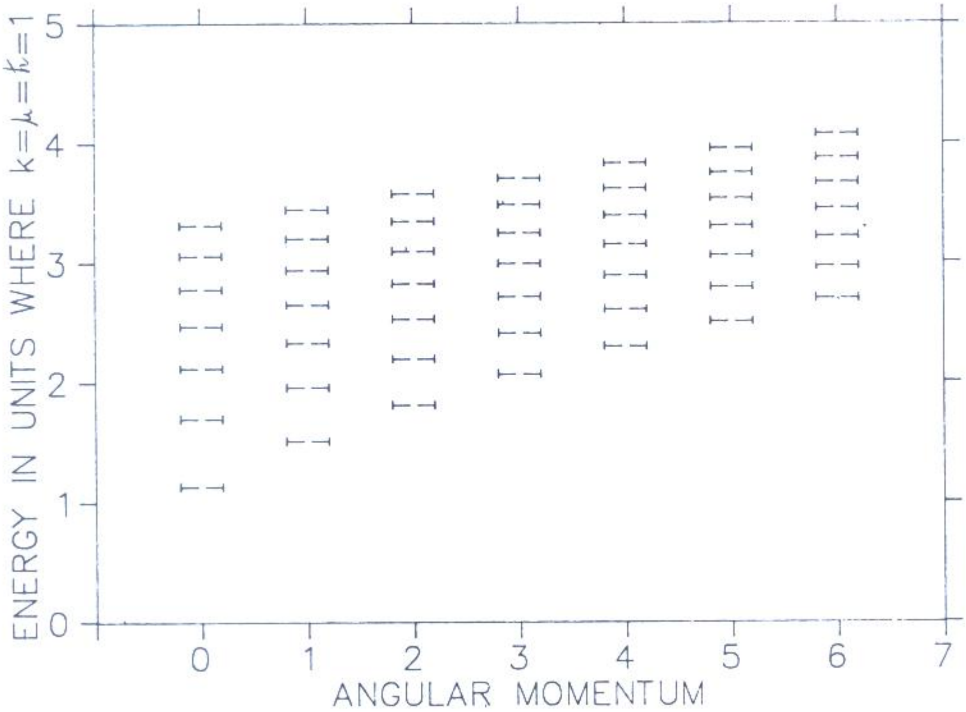

Figure 1. Plot of First-Order Linear Corrections Assuming a Background Three-Dimensional Oscillator Potential.

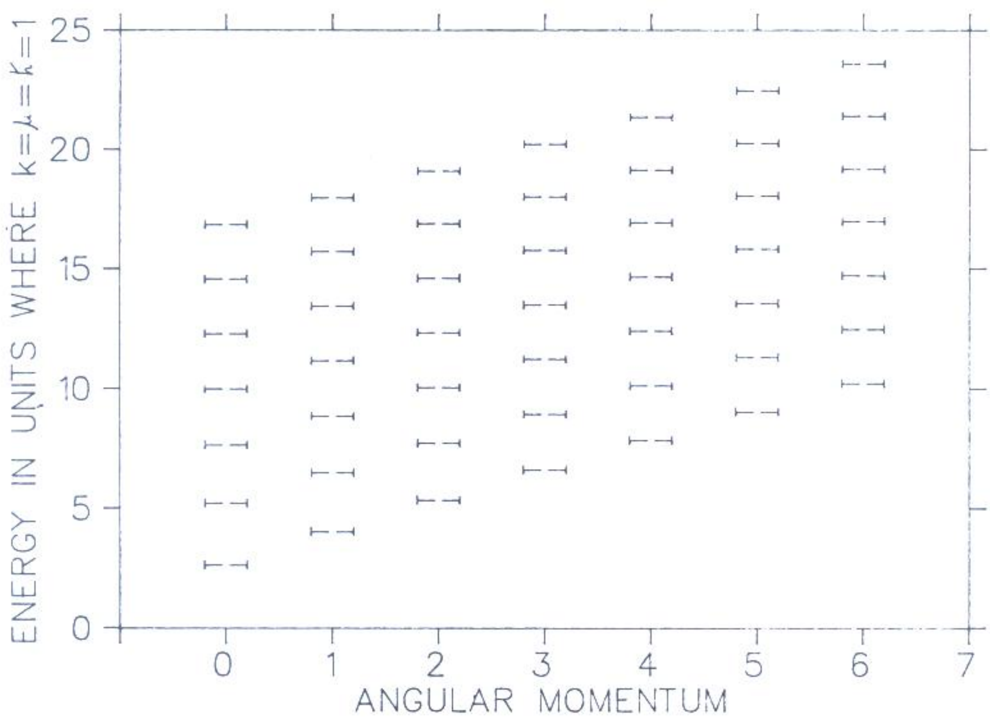

Figure 2. Plot of Energy to First-Order Assuming a Linear Perturbation and a Background Three-Dimensional Oscillator Potential.

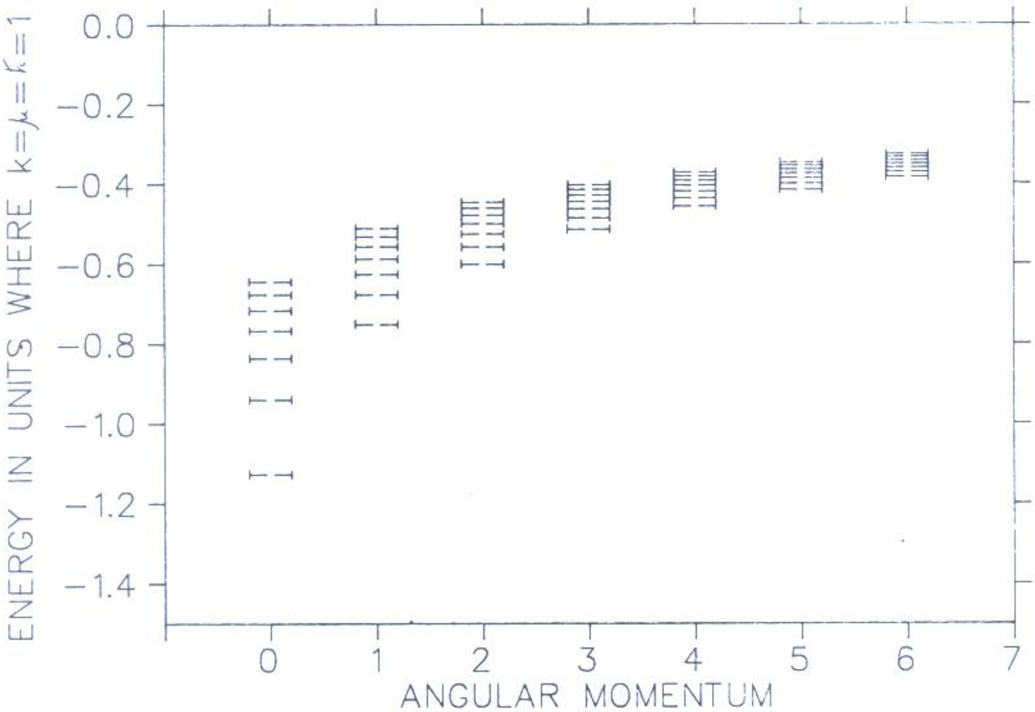

Figure 3. Plot of First-Order Coulomb-Type Corrections Assuming a Background Three-Dimensional Oscillator Potential.

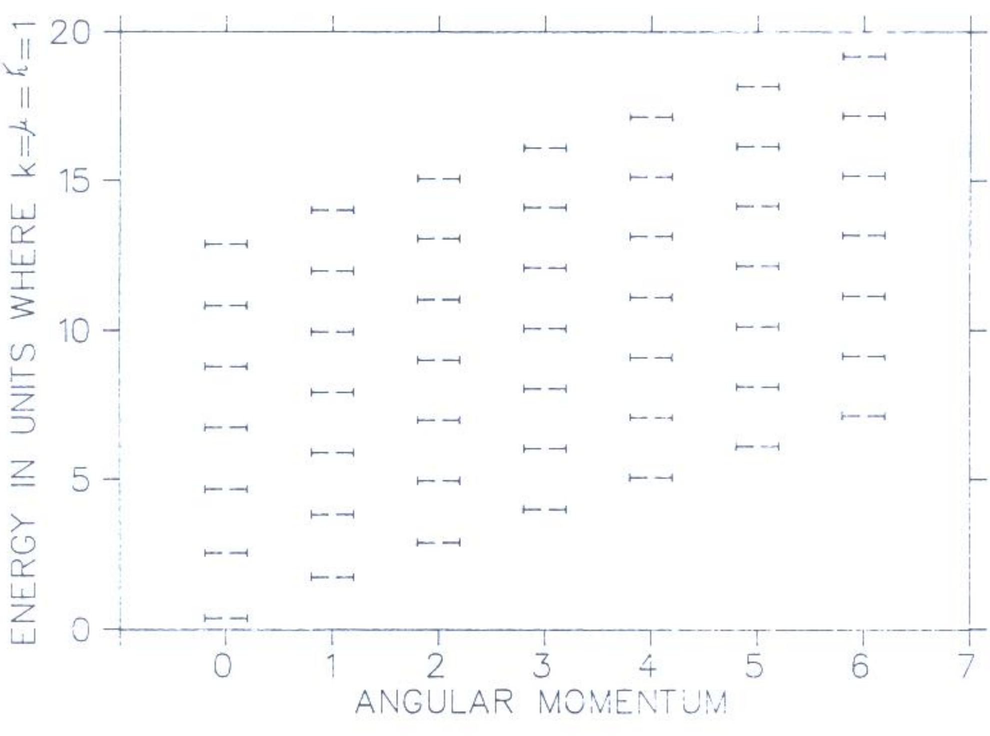

Figure 4. Plot of Energy to First-Order Assuming a Coulomb-Type Perturbation and a Background Three-Dimensional Oscillator Potential.

and using Equation (1) one obtains:

$$\Delta E_{nl}\left(\frac{\lambda}{r^{1+\varepsilon}}\right)=\frac{\lambda\Gamma(l\mp\varepsilon/2+1)\Gamma(n+1/2\pm\varepsilon/2)}{n!\Gamma(1/2\pm\varepsilon/2)\Gamma(l+3/2)}$$
$$\times\, {}_3F_2(-n,l+1\mp\varepsilon/2,1/2\mp\varepsilon/2;l+3/2,1/2\mp\varepsilon/2-n;1). \tag{12}$$

As $\varepsilon\to 0$ this result reduces to Equation (7). If $\varepsilon\to 2l+2$, substituting in Equation (12) one obtains:

$$\Delta E_{nl}(\lambda r^{2l+1})$$
$$=\frac{\lambda\Gamma(2l+2)\Gamma(n-l-1/2)}{n!\Gamma(-l-1/2)\Gamma(l+3/2)}$$
$$\times\, {}_3F_2(-n,2l+2,l+3/2;l+3/2,l+3/2-n;1)$$
$$=\frac{\lambda\Gamma(2l+2)\Gamma(n-l-1/2)\Gamma(l+3/2-n)}{n!\Gamma^2(l+3/2)\Gamma(-l-1/2-n)}$$
$$=\frac{\lambda 2^{2l+1}l!\Gamma(n-l-1/2)\Gamma(l+3/2-n)}{n!\sqrt{\pi}\Gamma(l+3/2)\Gamma(-l-1/2-n)}$$
$$=\frac{\lambda 2^{2l+1}l!\Gamma(n+l+3/2)}{n!\sqrt{\pi}\Gamma(l+3/2)}$$

i.e.

$$\Delta E_{n0}(\lambda r)=\frac{4\lambda\Gamma(n+3/2)}{\pi n!}\quad \text{if } l=0, \text{(see Equation (5))},$$

$$\Delta E_{0l}(\lambda r^{2l+1})=\frac{\lambda 2^{2l+1}l!}{\sqrt{\pi}}\quad \text{if } n=0,$$

etc.

SUMMARY

In this paper the boundary condition, that at large distances quarks are confined by a three-dimensional harmonic oscillator potential, is assumed. The first-order effect, on this system, of a linear and Coulomb potential are then obtained explicitly, as well as expressions for adjacent levels, and arbitrary l. Finally these results are plotted and the trends obtained are analyzed.

APPENDIX

In reference [2] we give *seven* special cases where ${}_3F_2$ in Equation (1) simplifies. Using contiguity relations [4] between ${}_3F_2$'s we obtain *two* additional special cases where the ${}_3F_2$ in Equation (1) simplify, namely:

$$\int_0^\infty \exp(-x)x^{a+1}L_n^a(x)L_{n'}^{a'}(x)\,dx$$
$$=\frac{\Gamma^2(n+a+1)\Gamma(n'+a'+1)\Gamma(n'-a+a'-1)}{n'!n!\Gamma(a'-a-1)}$$
$$\times\frac{\Gamma(a-a'-n'+2)\Gamma(-n'-1+n)}{\Gamma(a-a'-n'+n+2)\Gamma(-n'-1)}$$
$$\times\left\{(a-a'+2)+\frac{(a'-1)\,(n'-n+1)}{n'+1}\right\}.$$
$$=\frac{\Gamma^3(n+a+1)}{n!}(2n+a+1),\ (\text{if } n=n',\ a=a'), \tag{A.1}$$

the diagonal case ($n=n'$, $a=a'$) being a known result [8], and:

$$\int_0^\infty \exp(-x)x^{a+a'+1}L_n^a(x)L_{n'}^{a'}(x)\,dx$$
$$=\frac{\Gamma^2(n+a+1)\Gamma(n'+a'+1)\Gamma(a+a'+2)}{n'!n!\Gamma(-a-1)}$$
$$\times\frac{\Gamma(n'-a-1)\Gamma(a-n'+2)\Gamma(-n+n'-a'-1)}{\Gamma(a+2)\Gamma(a+2-n'+n)\Gamma(-n'-a'-1)}$$
$$\times\left\{(a+a'+2)+\frac{(a'+1)\,(-n'+n-a'-1)}{n'+a'+1}\right\}.$$
$$=\frac{\Gamma^4(n+a+1)\Gamma(2a+2)}{n!^2\Gamma(a+2)\Gamma(a+1)}(2n+a+1),$$
$$(\text{if } n=n',\ a=a'). \tag{A.2}$$

The contiguity relation required for (A.1) is obtained by writing

$$
\begin{aligned}
&{}_3F_2(-n, a+2, a-a'+2; a+1, a-a'+2-n'; 1) \\
&= \sum_{m=0}^{n} \frac{\Gamma(-n+m)\Gamma(a+2+m)\Gamma(a-a'+2+m)}{\Gamma(-n)\Gamma(a+2)\Gamma(a-a'+2)\Gamma(a+1+m)} \\
&\quad \times \frac{\Gamma(a+1)\Gamma(a-a'+2-n')}{\Gamma(a-a'+2-n'+m)} \frac{1}{m!} \\
&= \sum_{m=0}^{n} \frac{\Gamma(-n+m)\Gamma(a-a'+2+m)\Gamma(a-a'+2-n')}{\Gamma(-n)\Gamma(a-a'+2)\Gamma(a-a'+2-n'+m)} \\
&\quad \times \left\{\frac{a-a'+2}{a+1}\left(\frac{a-a'+2+m}{a-a'+2}\right) + \frac{a'-1}{a+1}\right\} \frac{1}{m!} \\
&= \frac{a-a'+2}{a+1} \\
&\quad \times {}_2F_1(-n, a-a'+3; a-a'+2-n'; 1) \\
&\quad + \frac{a'-1}{a+1}\, {}_2F_1(-n, a-a'+2; a-a'+2-n'; 1),
\end{aligned}
$$

which in turn may be simplified using Gauss's theorem Equation (4). A similar expression is

REFERENCES

[1] Fayyazuddin, "Perturbation Around Harmonic Oscillator Potential", *Arabian Journal for Science and Engineering*, accepted for publication.

[2] H. A. Mavromatis, *Intern. J. Computer Math.*, **36** (1990), p. 257.

[3] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products, 4th (English) Edn.* Prepared by A. Jeffrey. New York: Academic Press, 1980, p. 1045.

[4] L. J. Slater, *Generalized Hypergeometric Functions*. Cambridge University Press, 1960, p. 13.

[5] J. P. Elliott, A. D. Jackson, H. A. Mavromatis, E. A. Sanderson, and B. Singh, "Matrix Elements of the Nucleon–Nucleon Potential for Use in Nuclear-Structure Calculations", *Nucl. Phys. A*, **121** (1968), p. 241.

[6] I. N. Sneddon, *Special Functions of Mathematical Physics and Chemistry, 3rd Edn.* London: Longman, 1980, p. 37.

[7] L. J. Slater, *Generalized Hypergeometric Functions*. Cambridge University Press, 1960, p. 49.

[8] J. Powell and B. Crassman, *Quantum Mechanics*. Addison Wesley, 1965, p. 225.

[9] This procedure was recommended by one of the referees who reviewed this paper.

[10] I. N. Sneddon, Reference [6], p. 56.

[11] A. Erdelyi *et al.*, *Higher Transcendental Functions*. New York: McGraw–Hill, 1953, p. 3.

[12] I. N. Sneddon, Reference [6], p. 32.