Nonequilibrium relaxation of Bose-Einstein condensates: Real-time equations of motion and Ward identities

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Abstract

We present a field-theoretical method to obtain consistently the equations of motion for small amplitude condensate perturbations in a homogeneous Bose-condensed gas directly in real time. It is based on linear response, and combines the Schwinger-Keldysh formulation of nonequilibrium quantum field theory with the Nambu-Gor'kov formalism of quasiparticle excitations in the condensed phase and the tadpole method in quantum field theory. This method leads to causal equations of motion that allow to study the nonequilibrium evolution as an initial value problem. It also allows to extract directly the Ward identities, which are a consequence of the underlying gauge symmetry and which in equilibrium lead to the Hugenholtz-Pines theorem. An explicit one-loop calculation of the equations of motion beyond the Hartree-Fock-Bogoliubov approximation reveals that the nonlocal, absorptive contributions to the self-energies corresponding to the Beliaev and Landau damping processes are necessary to fulfill the Ward identities *in* or *out* of equilibrium. It is argued that a consistent implementation at low temperatures must be based on the loop expansion, which is shown to fulfill the Ward identities order by order in perturbation theory.

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1 Introduction

The realization of Bose-Einstein condensation (BEC) [1–4] in dilute atomic gases in which the atoms are confined in magnetic traps and cooled down to temperatures of order of fraction of microkelvins has opened a new era in atomic physics. The experimental and theoretical effort on the understanding of BEC is truly interdisciplinary [5] and the beautiful demonstration of BEC in atomic gases has rekindled both the experimental and theoretical interests [4].

The theoretical description of Bose-Einstein condensation in weakly interacting dilute gases has a long history [6–10] and has accounted for many experimental results [1–4]. Pioneering work on the microscopic theory of a Bose-condensed gas was done by Bogoliubov [11], whose original observation that the wave function of the excitations around the condensate (phonons) is a linear superposition of free particle states with equal and opposite momenta has been recently brilliantly confirmed experimentally [12]. A detailed and comprehensive theoretical description of the homogeneous and inhomogeneous BEC is offered in Refs. [6–9,5].

The current experiments on trapped Bose-condensed gases can measure with great accuracy the dynamical aspects of the collective (quasiparticle) excitations (see Ref. [4] for a thorough discussion). The earlier experiments [1–3] were carried at low temperatures and the measurements of the dynamics of the condensate revealed almost undamped oscillations with frequencies in excellent agreement with theoretical predictions based on the time dependent Gross-Pitaevskii equation [13,6–9,14,15].

More recent experiments [3,4] probe the low energy dynamics of collective excitations at higher temperatures and show evidence for large frequency shifts and strong damping. There are several important processes that contribute to damping of collective excitations. At zero temperature a quasiparticle can decay into two or more quasiparticles of lower energies, a mechanism that was originally studied by Beliaev [16] at zero temperature and by Popov [17] at finite temperature in a homogeneous Bose gas. At finite temperature an important damping mechanism in the collisionless regime is the Landau damping [18–20], which has recently been studied for trapped Bose gases [15] within a time dependent mean-field approximation.

The necessity for a firm theoretical understanding of the dynamical aspects of low-lying collective excitations in the condensed phase is underscored by the intense experimental effort to probe them. For a homogeneous Bose gas in the condensed phase a classification scheme for the different types of theoretical

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approximations to study the excitations has been put forth in a seminal article by Hohenberg and Martin [21] and more recently discussed in detail by Griffin [6,14,8]. A very important ingredient for a consistent description of the dynamics of quasiparticle excitations is the Hugenholtz-Pines theorem [22], which is a consequence of the underlying gauge invariance [21] and similar to the Goldstone theorem in that it guarantees gapless single (quasi)particle excitations if a continuous symmetry is broken with short range forces [23]. Since the Hugenholtz-Pines theorem is a consequence of the original gauge invariance and the Ward identities that stem from it, it is important to guarantee that any approximation scheme to study the dynamics of low energy excitations respects this theorem [14]. Detailed studies of the equations of motion for the small amplitude perturbation in the condensed phase [14,15] highlight the necessity for a systematic treatment compatible with the Hugenholtz-Pines theorem. The usual approach to obtaining the equations of motion begins with taking the expectation value of the Heisenberg equations of motion and invoking some type of factorization of the nonlinear correlators (see, e.g., Refs. [14,15] and references therein). It is at this point where approximation schemes to treat the nonlinear correlator may potentially conflict with the Hugenholtz-Pines theorem and the Ward identities.

The focus and goals. We focus on studying the real-time evolution of the condensate perturbations in a weakly interacting, homogeneous Bose-condensed gas to begin with, but our ultimate goals are to study consistently the nonequilibrium evolution of collective excitations and to establish contact with the experimental effort. In particular, we seek to obtain the oscillation frequencies and damping rates of condensate perturbation both for the homogeneous as well as for the inhomogeneous (trapped) cases and to study the nonequilibrium aspects of quasiparticle excitations in a Bose-condensed gas.

While the study of the homogeneous Bose gas may not be directly related to the experimental effort, a deeper understanding of nonequilibrium phenomena in many-body systems is intrinsically of fundamental importance and may pave the way to a systematic treatment of similar effects in trapped atomic gases. Of particular relevance for our study, at least for the homogeneous Bose gas, is to assess the impact of the infrared divergences in the self-energies on the frequencies and damping rates of collective excitations. In a strict perturbative expansion in the weakly interacting homogeneous Bose gas, the quasiparticle self-energies feature infrared divergences, logarithmic at zero temperature and power law at finite temperature [6,8]. Early work by Gavoret and Nozières [24], Nepomnyashchy and Nepomnyashchy [25] and Popov [26,27] points out that a resummation of the perturbative expansion leads to a cancellation of the infrared divergences in physical quantities. This result in turn suggests that the perturbative expansion requires a resummation to extract the quasiparticle frequencies and damping rates unambiguously, but the resummation scheme invoked should fulfill the Hugenholtz-Pines theorem. To our knowledge these issues have not been so thoroughly studied and whether such divergences require novel treatment in the case of trapped atomic gases seems to be an open question.

In this article we propose a consistent formulation of the nonequilibrium dynamics by implementing a field-theoretical approach to study dynamics out of equilibrium. The field-theoretical approach to BEC has been clearly formulated both in the operator [28] as well as in the path integral formulation [27,29]. The functional integral formulation is particularly convenient as it can be generalized to study nonequilibrium phenomena by implementing the Schwinger-Keldysh formulation [30–35] at the level of the path integrals. Recently, Stoof [36] has advanced the Schwinger-Keldysh formulation of nonequilibrium field theory to study the dynamics of condensation in Bose-condensed gases.

Our ultimate goal is to obtain a consistent and systematic analysis of quasiparticle frequencies and damping rates in both the homogeneous and inhomogeneous Bose-condensed gases from a nonequilibrium approach. We begin such program in this article by establishing a nonequilibrium framework that allows to extract this information directly from the real-time evolution of small amplitude perturbations around the condensate in a homogeneous Bose-condensed gas.

Brief summary of results. The main results of this article, which we consider the first in a program that is devoted to setting up the nonequilibrium formulation, are the summarized as follows.

(i) We combine the Schwinger-Keldysh nonequilibrium formulation [30–35] and the Nambu-Gor'kov formalism [6,8,28,37] for treating the Bogoliubov quasiparticles in the condensed phase, along with a novel method [38] introduced within the context of quantum field theory, to obtain the equations of motion for small amplitude perturbations away from equilibrium around the condensate *directly in real time*. This method leads to causal equations of motion and allows to study the evolution of small amplitude perturbations of the condensate as an *initial value problem* [39] which describes an experimental situation.

(ii) The novel method [38] implemented to obtain directly the equations of motion in real time leads to a simple derivation of the Ward identities associated with the underlying gauge symmetry *in* or *out* of equilibrium. In equilibrium these Ward identities lead directly to the Hugenholtz-Pines theorem. This method is therefore seen to lead to consistent equations of motion for quasiparticle excitations of the condensate, that fulfill the Hugenholtz-Pines theorem when the homogeneous condensate is in equilibrium. We show explicitly to one-loop order that the inclusion of absorptive contributions beyond the Hartree-Fock-Bogoliubov approximation are necessary to fulfill the

Ward identities, which, when the condensate is in equilibrium become the Hugenholtz-Pines theorem. We obtain explicitly the one-loop self-energies including the absorptive contributions and their spectral representations beyond the Hartree-Fock-Bogoliubov approximation.

(iii) We discuss nonequilibrium aspects such as the instabilities in the quasiparticle spectrum when the homogeneous condensate is not in equilibrium. We also discuss subtleties associated with the perturbative expansion that require a consistent rearrangement of the perturbative series in a *loop* expansion, not in an expansion in the bare coupling constant.

The article is organized as follows. In Sec. 2 we introduce the model and the linear response formulation to obtain the equations of motion. In Sec. 3 we introduce the Schwinger-Keldysh formulation in the Nambu-Gor'kov formalism to study the nonequilibrium aspects of Bogoliubov quasiparticles. In section 4 we introduce the tadpole method, obtain the equations of motion directly in real time and cast them in terms of an initial value problem. We obtain explicitly the retarded self-energies up to one-loop order and obtain their spectral representations. In Sec. 5 we derive the generalized Ward identities, present an alternative derivation of the Hugenholtz-Pines theorem and confirm that the one-loop equations of motion obtained do fulfill this theorem and highlight the necessity for including the absorptive parts of the self-energies. In Sec. 6 we highlight relevant aspects of the perturbative expansions and describe the condensate instabilities featured in the quasiparticle spectrum when the condensate is away from equilibrium. Section 7 presents our conclusions and poses new questions. An appendix is devoted to an alternative derivation of the Bogoliubov transformation, which facilitates the Schwinger-Keldysh nonequilibrium formulation.

2 Preliminaries

2.1 The model

The Hamiltonian of a Bose gas in the absence of a trapping one-body potential is given by

$$
\begin{aligned}
H = & \int d^3x\, \psi^\dagger(\mathbf{x},t) \left(-\frac{\nabla^2}{2m}\right) \psi(\mathbf{x},t) \\
& + \frac{g}{2} \int d^3x\, \psi^\dagger(\mathbf{x},t)\psi^\dagger(\mathbf{x},t)\psi(\mathbf{x},t)\psi(\mathbf{x},t),
\end{aligned}
\tag{2.1}
$$

where $\psi(\mathbf{x},t)$ is the Heisenberg complex scalar field representing spinless bosons of mass m, $g = 4\pi a/m$ is the strength of the pseudopotential with $a > 0$ being the s-wave scattering length and we have set $\hbar = k_B = 1$. We will recover $\hbar$ in Sec. 6, where we analyze the nature of the loop expansion. The field $\psi(\mathbf{x}, t)$ and its Hermitian conjugate satisfy equal-time commutation relations

$$[\psi(\mathbf{x}, t), \psi^\dagger(\mathbf{x}', t)] = \delta^{(3)}(\mathbf{x} - \mathbf{x}'),$$
$$[\psi(\mathbf{x}, t), \psi(\mathbf{x}', t)] = [\psi^\dagger(\mathbf{x}, t), \psi^\dagger(\mathbf{x}', t)] = 0. \tag{2.2}$$

A more systematic treatment of interactions requires a T-matrix formulation [36] but in order to simplify the discussion we will only consider the local s-wave interaction at this stage. It will become clear that the method can be straightforwardly generalized to include the T-matrix resummation of the interaction.

The Hamiltonian H is invariant under the $U(1)$ gauge transformation

$$\psi(\mathbf{x}, t) \to e^{i\theta}\psi(\mathbf{x}, t),$$
$$\psi^\dagger(\mathbf{x}, t) \to e^{-i\theta}\psi^\dagger(\mathbf{x}, t), \tag{2.3}$$

where θ is a constant phase. A consequence of this $U(1)$ gauge symmetry is conservation of the number of particles. Indeed, the number operator of particles

$$N = \int d^3x \, \psi^\dagger(\mathbf{x}, t)\psi(\mathbf{x}, t) \tag{2.4}$$

commutes with H and hence is a constant of motion. However, it is convenient to work in the grand-canonical ensemble in which the grand-canonical Hamiltonian is given by

$$\begin{aligned} K \equiv\, & H - \mu N \\ =\, & \int d^3x \, \psi^\dagger(\mathbf{x}, t)\left(-\frac{\nabla^2}{2m} - \mu\right)\psi(\mathbf{x}, t) \\ & + \frac{g}{2}\int d^3x \, \psi^\dagger(\mathbf{x}, t)\psi^\dagger(\mathbf{x}, t)\psi(\mathbf{x}, t)\psi(\mathbf{x}, t), \end{aligned} \tag{2.5}$$

where the chemical potential μ is the Lagrange multiplier associated with conservation of number of particles. The corresponding Lagrangian (density) is given by

$$\mathcal{L}[\psi^\dagger, \psi] = \psi^\dagger\left(\frac{i}{2}\frac{\overleftrightarrow{\partial}}{\partial t} + \frac{\nabla^2}{2m} + \mu\right)\psi - \frac{g}{2}\psi^\dagger\psi^\dagger\psi\psi, \tag{2.6}$$

where $\frac{\overleftrightarrow{\partial}}{\partial t} = \frac{\overrightarrow{\partial}}{\partial t} - \frac{\overleftarrow{\partial}}{\partial t}$.

In a Bose-condensed gas, the condensate plays a crucial role [11] and hence it is convenient to decompose the field into the condensate and noncondensate parts [14]

$$\psi(\mathbf{x},t) = \phi(\mathbf{x},t) + \chi(\mathbf{x},t), \quad \langle \chi(\mathbf{x},t) \rangle = 0, \tag{2.7}$$

where $\phi(\mathbf{x},t) \equiv \langle \psi(\mathbf{x},t) \rangle$ is referred to in the literature as the condensate wave function and $\chi(\mathbf{x},t)$ is the noncondensate operator. In the above expression, $\langle \mathcal{O}(\mathbf{x},t) \rangle = \mathrm{Tr}[\rho \mathcal{O}(\mathbf{x},t)]/\mathrm{Tr}\rho$ denotes the *expectation value* of the Heisenberg operator $\mathcal{O}(\mathbf{x},t)$ in the *initial density matrix* ρ. The presence of the condensate $\phi(\mathbf{x},t) \neq 0$ leads to spontaneous breaking of the $U(1)$ gauge symmetry and results in a profound consequence for the spectrum of the quasiparticle excitations.

In the absence of trapping potentials and explicit symmetry breaking external sources, the condensate is homogeneous (i.e., space-time independent) and denoted by $\phi(\mathbf{x},t) = \phi_0$, which is the situation under consideration in this article.

2.2 Real-time relaxation in linear response

The goal in this article is to obtain *directly in real time* the equations of motion for small amplitude perturbations of the homogeneous condensate in an initial value problem formulation. Our strategy to study the relaxation of the condensate perturbation as an initial value problem begins with preparing a Bose-condensed gas slightly perturbed away from equilibrium by applying external source coupled to the field. Once the external source is switched off, the perturbed condensate must relax towards equilibrium. It is precisely this *real-time evolution* of the nonequilibrium condensate relaxation that we aim to study in this article.

Let $\eta(\mathbf{x},t)$ be an external c-number source coupled to the quantum field $\psi(\mathbf{x},t)$, then the Lagrangian given by (2.6) becomes

$$\mathcal{L}[\psi^\dagger,\psi] \to \mathcal{L}[\psi^\dagger,\psi] + \psi^\dagger \eta + \eta^* \psi. \tag{2.8}$$

The presence of external source will induce a (linear) response of the system in the form of an induced expectation value

$$\langle \psi(\mathbf{x},t) \rangle_\eta = \phi_0 + \delta(\mathbf{x},t), \tag{2.9}$$

where $\langle \mathcal{O}(\mathbf{x},t) \rangle_\eta$ denotes the expectation value of $\mathcal{O}(\mathbf{x},t)$ in the presence of external source and $\delta(\mathbf{x},t)$ is the space-time dependent perturbation of the homogeneous condensate ϕ_0 *induced by the external source*. The linear response perturbation $\delta(\mathbf{x},t)$ vanishes when the external source $\eta(\mathbf{x},t)$ vanishes at all times. This is tantamount to decomposing the field into the homogeneous condensate (ϕ_0), a small amplitude perturbation induced by the external source

$[\delta(\mathbf{x},t)]$, and the noncondensate part $[\chi(\mathbf{x},t)]$ as

$$\psi(\mathbf{x},t)=\phi_0+\delta(\mathbf{x},t)+\chi(\mathbf{x},t), \quad \langle\chi(\mathbf{x},t)\rangle_\eta=0. \tag{2.10}$$

In linear response theory $\delta(\mathbf{x},t)$ can be expressed in terms of the *exact* retarded Green's function of the field in the absence of external source [28,39]. An experimentally relevant initial value problem formulation for the real-time relaxation of the condensate perturbation can be obtained by considering that the external source is adiabatically switched on at $t=-\infty$ and switched off at $t=0$, i.e.,

$$\eta(\mathbf{x},t)=\eta(\mathbf{x})\,e^{\epsilon t}\,\Theta(-t), \quad \epsilon\to 0^+. \tag{2.11}$$

The adiabatic switching-on of the external source induces a space-time dependent condensate perturbation $\delta(\mathbf{x},t)$, which is prepared adiabatically by the external source with a given value $\delta(\mathbf{x},0)$ at $t=0$ determined by $\eta(\mathbf{x})$. For $t>0$ after the external source has been switched off, the perturbed condensate will evolve in the absence of any external source relaxing towards equilibrium. Thus, the external source $\eta(\mathbf{x},t)$ is necessary for preparing an initial state at $t=0$ and setting up an initial value problem. This method has been applied to study a variety of damping and relaxation phenomena in relativistic hot and dense plasmas [39] and damping of photons in a strong magnetic field [40].

Using the decomposition (2.10) and consistent with linear response by keeping only the linear terms in δ and δ^*, the Lagrangian $\mathcal{L}$ becomes (in the presence of external source)

$$\mathcal{L}[\chi^\dagger,\chi]=\mathcal{L}_0[\chi^\dagger,\chi]+\mathcal{L}_{\rm int}[\chi^\dagger,\chi], \tag{2.12}$$

with

$$\begin{aligned}
\mathcal{L}_0[\chi^\dagger,\chi]=&\,\chi^\dagger\left(\frac{i}{2}\frac{\overleftrightarrow{\partial}}{\partial t}+\frac{\nabla^2}{2m}+\mu-2g|\phi_0|^2\right)\chi-\frac{g}{2}\left(\phi_0^2\chi^\dagger\chi^\dagger+\phi_0^{*2}\chi\chi\right),\\
\mathcal{L}_{\rm int}[\chi^\dagger,\chi]=&\,\chi^\dagger\bigg[\left(i\frac{\partial}{\partial t}+\frac{\nabla^2}{2m}+\mu-2g|\phi_0|^2\right)\delta-g\phi_0^2\delta^*\\
&+\phi_0\left(\mu-g|\phi_0|^2\right)+\eta\bigg]-2g\phi_0^*\delta\chi^\dagger\chi-g\phi_0\delta\chi^\dagger\chi^\dagger\\
&-g\phi_0\chi\chi^\dagger\chi^\dagger-g\delta\chi\chi^\dagger\chi^\dagger-\frac{g}{2}\chi^\dagger\chi^\dagger\chi\chi+\text{H.c.},
\end{aligned} \tag{2.13}$$

where we have discarded the c-number (field operators independent) and surface terms. Consistent with linear response, we have only kept linear terms in δ, δ^*, which are the small amplitude departure from the homogeneous condensate induced by the external source η. We note that the Lagrangian $\mathcal{L}[\chi^\dagger,\chi]$

is obviously invariant under the gauge transformations

$$\begin{aligned} \phi_0, \delta, \chi, \eta &\to e^{i\theta}\phi_0, e^{i\theta}\delta, e^{i\theta}\chi, e^{i\theta}\eta, \\ \phi_0^*, \delta^*, \chi^\dagger, \eta^\dagger &\to e^{-i\theta}\phi_0^*, e^{-i\theta}\delta^*, e^{-i\theta}\chi^\dagger, e^{-i\theta}\eta^\dagger, \end{aligned} \tag{2.14}$$

which, as will be seen below, is at the heart of the Ward identities that will lead to the Hugenholtz-Pines theorem.

Whereas in general a gauge transformation is invoked (correctly) to fix the condensate ϕ_0 to be *real* for convenience, this choice corresponds to *fixing a particular gauge*, which in turn hides the underlying gauge symmetry. In order to obtain the Ward identity associated with this symmetry we will keep a complex condensate ϕ_0 and analyze in detail the transformation laws of the various contributions to the equations of motion.

3 Nonequilibrium formulation

3.1 Generating functional

The general framework to study of nonequilibrium phenomena is the Schwinger-Keldysh formulation [30–35], which we briefly review here in a manner leads immediately to a path integral formulation. For an alternative presentation see, e.g., Refs. [36].

Consider that the system is described by an initial density matrix ρ and a perturbation is switched on at a time t_0, so that the total Hamiltonian for $t > t_0$, $H(t)$, does not commute with the initial density matrix. The expectation value of a Heisenberg operator $\mathcal{O}(t) = U^{-1}(t,t_0)\mathcal{O}(t_0)U(t,t_0)$ is given by

$$\langle \mathcal{O}(t) \rangle = \frac{\mathrm{Tr}\rho U^{-1}(t,t_0)\mathcal{O}U(t,t_0)}{\mathrm{Tr}\rho}, \tag{3.1}$$

where $U(t,t_0)$ is the unitary time evolution operator in the Heisenberg picture

$$U(t,t_0) = T \exp\left[-i\int_{t_0}^{t} dt'\, H(t')\right], \tag{3.2}$$

with T the time-ordering symbol. If the initial density matrix ρ describes a state in thermal equilibrium at inverse temperature β with the unperturbed Hamiltonian $H(t < t_0) = H$, i.e.,

$$\rho = e^{-\beta H} = U(t_0 - i\beta, t_0), \tag{3.3}$$

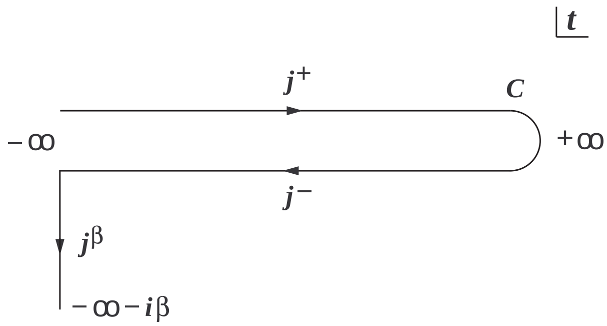

Fig. 1. The contour $\mathcal{C}$ in complex time plane in the Schwinger-Keldysh formulation. It consists of a forward branch running from $t = -\infty$ to $t = +\infty$, a backward branch from $t = +\infty$ back to $t = -\infty$, and an imaginary branch from $t = -\infty$ to $t = -\infty - i\beta$. The sources $j^{\pm}$ serve to generate the real-time nonequilibrium Green's functions.

then the expectation value (3.1) can be written in the form

$$\langle \mathcal{O}(t) \rangle = \frac{\mathrm{Tr} U(t_0 - i\beta, t_0) U^{-1}(t_0, t) \mathcal{O} U(t, t_0)}{\mathrm{Tr} U(t_0 - i\beta, t_0)}. \tag{3.4}$$

The numerator of this expression has the following interpretation: evolve in time from t_0 up to t, insert the operator $\mathcal{O}$, evolve back from t to the initial time t_0 and down the imaginary axis in time from t_0 to $t_0 - i\beta$. The denominator describes the evolution in imaginary time which is the familiar description of a thermal density matrix. We note that unlike the S-matrix elements or transition amplitudes, expectation values of Heisenberg operators require evolution *forward* and *backward* in time (corresponding to the U and U^{-1} on each side of the operator $\mathcal{O}$).

The time evolution operators have a path-integral representation [41] in terms of the Lagrangian, and the insertion of operators can be systematically handled by introducing sources coupled linearly to the field operators [43], i.e.,

$$\mathcal{L}[\chi^\dagger, \chi] \to \mathcal{L}[\chi^\dagger, \chi] + j^* \chi + \chi^\dagger j. \tag{3.5}$$

The introduction of sources also allows a systematic perturbative expansion, since in such an expansion, powers of operators are obtained by functional derivatives with respect to these sources, which are set to zero after functional differentiation. We note that these sources j, j^* introduced to generate the perturbative expansion in terms of functional derivatives with respect to these, are *different* from the external sources η, η^* introduced in (2.8) to generate an initial value problem in linear response and to displace the condensate from equilibrium.

Since there are *three* different time evolution operators, the forward, backward and imaginary, we introduce *three* different sources for each one of these time evolution operators, respectively. Taking $t_0 \to -\infty$, we are led to considering the generating functional [38,39,42]

$$Z[j^+, j^-, j^\beta] = \mathrm{Tr} U(-\infty - i\beta, -\infty, j^\beta) U(-\infty, +\infty, j^-) U(+\infty, -\infty; j^+), \tag{3.6}$$

where $U(t_f, t_i; j)$ is the time evolution operator [see (3.2)] in the presence of the source j and for simplicity of notation we have not displayed the complex conjugate of the sources j^*. The denominator in (3.4) is given by $\mathrm{Tr}\rho = Z[0,0,0]$. The generating functional $Z[j^+, j^-, j^\beta]$ can be written as a path integral along the contour in (complex) time plane (see Fig. 1)

$$Z[j^+, j^-, j^\beta] = \int \mathcal{D}_{\mathcal{C}} \chi^\dagger \mathcal{D}_{\mathcal{C}} \chi \, \exp \left[i \int_{\mathcal{C}} d^4x \, \mathcal{L}_{\mathcal{C}}[\chi^\dagger, \chi, j] \right], \tag{3.7}$$

where $\mathcal{D}_{\mathcal{C}} \chi^\dagger \mathcal{D}_{\mathcal{C}} \chi$ denotes the functional integration measure along the contour $\mathcal{C}$ and

$$\begin{aligned} \int_{\mathcal{C}} d^4x \, \mathcal{L}_{\mathcal{C}}[\chi^\dagger, \chi, j] \equiv & \int_{-\infty}^{+\infty} d^4x \, \mathcal{L}[\chi^{\dagger +}, \chi^+, j^+] - \int_{-\infty}^{+\infty} d^4x \, \mathcal{L}[\chi^{\dagger -}, \chi^-, j^-] \\ & + \int_{-\infty}^{-\infty - i\beta} d^4x \, \mathcal{L}[\chi^{\dagger \beta}, \chi^\beta, j^\beta]. \end{aligned} \tag{3.8}$$

with $\int_{-\infty}^{+\infty} d^4x \equiv \int d^3x \int_{-\infty}^{+\infty} dt$, etc. Because of the trace and the bosonic nature of the operators, the path integral along the contour $\mathcal{C}$ requires *periodic boundary conditions* on the fields. The superscripts $+$ and $-$ refer to fields defined in the upper and lower branches, respectively, corresponding to forward $(+)$ and backward $(-)$ time evolution, while the superscript β refers to the field defined in the vertical branch running down parallel to the imaginary axis. The negative sign in front of the action along the backward branch is a result of the fact that backward time evolution is determined by $U^{-1}(+\infty, -\infty)$ with U the time evolution operator. The contour source j that enters in the contour Lagrangian $\mathcal{L}_{\mathcal{C}}$ in (3.7) takes the values of the sources $j^\pm$ and j^β in the respective branches as displayed in Fig. 1.

Functional derivatives with respect to the sources in the forward branch give time-ordered products of operators, those with respect to the sources in the backward branch give the anti-time-ordered products of operators, and those with respect to the sources in the imaginary branch give the usual imaginary-time (Matsubara) correlation functions. While the sources j^+, j^- and j^β introduced to obtain the correlation functions via functional differentiation are *different* in the different branches, as they generate the time-ordered, anti-time-ordered and Matsubara correlation functions, respectively; the external source η, the homogeneous condensate ϕ_0, and the departure from equilibrium δ are c-numbers and hence treated the *same* in all branches.

Writing the Lagrangian as a free and an interaction part as $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}$, the generating functional can be written as a power series expansion in the interaction part, which in turn can be generated by taking functional derivatives with respect to the sources j, j^* by identifying

$$\begin{aligned} \chi^{\pm} \to \mp i\frac{\delta}{\delta j^{*\pm}}, \quad \chi^{\dagger\pm} \to \mp i\frac{\delta}{\delta j^{\pm}}, \\ \chi^{\beta} \to -i\frac{\delta}{\delta j^{*\beta}}, \quad \chi^{\dagger\beta} \to -i\frac{\delta}{\delta j^{\beta}}. \end{aligned} \tag{3.9}$$

As a result, the full generating functional along the contour $\mathcal{C}$ can be written as

$$Z[j] = \exp\left\{ i\int_{\mathcal{C}} d^4x\, \mathcal{L}_{\text{int},\mathcal{C}}\left[-i\frac{\delta}{\delta j^*}, -i\frac{\delta}{\delta j}\right]\right\} Z_0[j], \tag{3.10}$$

where free field generating functional $Z_0[j]$ is given by (3.7) with the non-interacting Lagrangian $\mathcal{L}_0[\chi^\dagger, \chi]$ given by (2.13).

3.2 Green's functions

The calculation of $Z_0[j]$ is facilitated by introducing the Nambu-Gor'kov formalism [37,28,6]. Let us introduce the (bosonic) two-component Nambu-Gor'kov fields

$$\Psi(\mathbf{x}, t) = \begin{bmatrix} \chi(\mathbf{x}, t) \\ \chi^\dagger(\mathbf{x}, t) \end{bmatrix}, \quad \Psi^\dagger(\mathbf{x}, t) = \left[\chi^\dagger(\mathbf{x}, t), \chi(\mathbf{x}, t)\right], \tag{3.11}$$

the corresponding sources

$$J(\mathbf{x}, t) = \begin{bmatrix} j(\mathbf{x}, t) \\ j^*(\mathbf{x}, t) \end{bmatrix}, \quad J^\dagger(\mathbf{x}, t) = \left[j^*(\mathbf{x}, t), j(\mathbf{x}, t)\right], \tag{3.12}$$

and the following 2×2 matrices

$$\sigma_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \sigma_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{3.13}$$

in terms of which the quadratic part of the Lagrangian given by (2.13) and now defined on the contour $\mathcal{C}$ in Fig. 1 can be written as

$$\mathcal{L}_0[\Psi^\dagger, \Psi] = \frac{1}{2}\Psi^\dagger\left[i\sigma_3\frac{\partial}{\partial t} + \frac{\nabla^2}{2m} + \mu - 2g|\phi_0|^2 - g\phi_0^2\sigma_+ - g\phi_0^{*2}\sigma_-\right]\Psi, \tag{3.14}$$

where the time derivative is understood to be taken with respect to the contour $\mathcal{C}$. The equation of motion for the free Nambu-Gor'kov field Ψ in the presence of the source J reads

$$\left[i\sigma_3 \frac{\partial}{\partial t} + \frac{\nabla^2}{2m} + \mu - 2g|\phi_0|^2 - g\phi_0^2\sigma_+ - g\phi_0^{*2}\sigma_- \right] \Psi(\mathbf{x}, t) = -J(\mathbf{x}, t). \quad (3.15)$$

The solution of this equation of motion is given by

$$\Psi_J(\mathbf{x}, t) = -\int_{\mathcal{C}} d^4x' G(\mathbf{x} - \mathbf{x}', t - t') J(\mathbf{x}', t'), \quad (3.16)$$

where $G(\mathbf{x} - \mathbf{x}', t - t')$ is the Green's function along this contour and satisfies

$$\left[i\sigma_3 \frac{\partial}{\partial t} + \frac{\nabla_{\mathbf{x}}^2}{2m} + \mu - 2g|\phi_0|^2 - g\phi_0^2\sigma_+ - g\phi_0^{*2}\sigma_- \right] \\ \times G(\mathbf{x} - \mathbf{x}', t - t') = \delta_{\mathcal{C}}(t - t')\, \delta^{(3)}(\mathbf{x} - \mathbf{x}'), \quad (3.17)$$

with $\delta_{\mathcal{C}}(t - t')$ the Dirac delta function along the contour $\int_{\mathcal{C}} dt'\, \delta_{\mathcal{C}}(t - t') = 1$. The Green's function $G(\mathbf{x} - \mathbf{x}', t - t')$ has the form

$$G(\mathbf{x} - \mathbf{x}', t - t') = G^{>}(\mathbf{x} - \mathbf{x}', t - t')\Theta_{\mathcal{C}}(t - t') + G^{<}(\mathbf{x} - \mathbf{x}', t - t')\Theta_{\mathcal{C}}(t' - t), \quad (3.18)$$

where $\Theta_{\mathcal{C}}(t - t')$ is the step function along the contour and $G^{\gtrless}(\mathbf{x} - \mathbf{x}', t - t')$ obey the homogeneous equations of motion.

The periodic boundary conditions on the fields in the path integral, a result of the trace over bosonic fields in (3.6), lead to the following boundary condition on the Green's function

$$\lim_{t_0 \to -\infty} G(\mathbf{x} - \mathbf{x}', t_0 - t') = \lim_{t_0 \to -\infty} G(\mathbf{x} - \mathbf{x}', t_0 - i\beta - t'). \quad (3.19)$$

Since along the contour $t_0 \to -\infty$ is the *earliest* time and $t_0 - i\beta$ is therefore the *latest* time, (3.19) entails

$$\lim_{t_0 \to -\infty} G^{<}(\mathbf{x} - \mathbf{x}', t_0 - t') = \lim_{t_0 \to -\infty} G^{>}(\mathbf{x} - \mathbf{x}', t_0 - i\beta - t'), \quad \forall\, t', \quad (3.20)$$

which is the Kubo-Martin-Schwinger (KMS) condition for equilibrium correlation functions [43].

The free field generating functional $Z_0[J]$ is now obtained by writing

$$\Psi(\mathbf{x}, t) = \widetilde{\Psi}(\mathbf{x}, t) + \Psi_J(\mathbf{x}, t), \\ \Psi^\dagger(\mathbf{x}, t) = \widetilde{\Psi}^\dagger(\mathbf{x}, t) + \Psi_J^\dagger(\mathbf{x}, t), \quad (3.21)$$

which leads to the result

$$Z_0[J] = Z_0[0] \exp\left[-\frac{i}{2}\int_{\mathcal{C}} d^4x \int_{\mathcal{C}} d^4x' J^\dagger(x) G(x-x') J(x')\right], \tag{3.22}$$

where and hereafter x denotes the space-time coordinates $(\mathbf{x}, t)$ for simplicity of notation. The source independent term $Z_0[0]$ will cancel between the numerator and the denominator in all expectation values in (3.1).

Furthermore, we are interested in computing correlation functions of *finite real times* which are defined for fields in the forward $(+)$ and backward $(-)$ time branches but not in the imaginary branch. For these real-time correlation functions the contributions to the generating functional from one source in the imaginary branch and another source in either the forward or backward branch vanish by the Riemann-Lebesgue lemma [33,38,42], since the time arguments are infinitely far apart along the contour. Therefore the contour integrals of the source terms and Green's functions in the generating functional factorize into a term in which the sources are those either in the forward and backward branches and another term in which *both* sources are in the imaginary branch [38,42]. The latter term (with both sources in the imaginary branch) cancel between numerator and denominator in expectation values and the only remnant of the imaginary branch is through the periodic boundary conditions along the full contour in the Green's function.

Thus the generating functional for real-time correlation functions simplifies to the following expression, defined solely along the forward and backward time branches [33–35,38,42],

$$\begin{aligned} Z[J^\pm, J^{\dagger\pm}] = \exp\Big[i\int_{-\infty}^{+\infty} d^4x \Big(&\mathcal{L}_{\text{int}}[-i\delta/\delta J^{*+}, -i\delta/\delta J^+] \\ &-\mathcal{L}_{\text{int}}[i\delta/\delta J^{*-}, i\delta/\delta J^-]\Big)\Big] \exp\Big\{-\frac{i}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} d^4x d^4x' \\ &\times\Big[J^{\dagger+}(x)G^{++}(x,x')J^+(x') + J^{\dagger-}(x)G^{--}(x,x')J^-(x') \\ &- J^{\dagger+}(x)G^{+-}(x,x')J^-(x') - J^{\dagger-}(x)G^{-+}(x,x')J^+(x')\Big]\Big\}, \end{aligned} \tag{3.23}$$

with

$$\begin{aligned} G^{++}(x,x') &= G^>(\mathbf{x}-\mathbf{x}', t-t')\Theta(t-t') + G^<(\mathbf{x}-\mathbf{x}', t-t')\Theta(t'-t), \\ G^{--}(x,x') &= G^>(\mathbf{x}-\mathbf{x}', t-t')\Theta(t'-t) + G^<(\mathbf{x}-\mathbf{x}', t-t')\Theta(t-t'), \\ G^{-+}(x,x') &= G^>(\mathbf{x}-\mathbf{x}', t-t'), \\ G^{+-}(x,x') &= G^<(\mathbf{x}-\mathbf{x}', t-t'), \end{aligned} \tag{3.24}$$

where now $-\infty \leq t, t' \leq +\infty$ and the superscripts $+$, $-$ correspond to the sources defined on the forward ($+$) and backward ($-$) time branches, respectively. An important issue that must be highlighted at this stage, is that derivatives with respect to sources in the forward ($+$) time branch correspond to insertion of operators *pre-multiplying* the density matrix ρ and derivatives with respect to sources in the backward ($-$) branch correspond to the insertion of operators *post-multiplying* the density matrix. That this is so is a consequence of the fact that the density matrix evolves in time as $U(t, t_0)\rho_0 U^{-1}(t, t_0)$ with $U(t, t_0)$ the time evolution operator.

These four correlation functions are not independent because of the identity

$$G^{++}(x, x') + G^{--}(x, x') - G^{+-}(x, x') - G^{-+}(x, x') = 0. \tag{3.25}$$

The diagonal elements in $G^{++}(x, x')$ are the normal Green's functions, representing the propagation of single noncondensate particles, whereas the off-diagonal elements are the anomalous Green's functions, corresponding to the annihilation and creation of two noncondensate particles, respectively.

The functions $G^{\gtrless}(x, x')$, which are solutions of the homogeneous free field equation of motion, are simply related to the correlation functions of the free Nambu-Gor'kov fields Ψ, $\Psi^\dagger$. Indeed, taking variational derivatives of the free field generating functional $Z_0[J]$ with respect to $j^{*\pm}$ and $j^\pm$ that make up the source $J^\pm$, one can easily show that

$$\begin{aligned} G^{>}(x, x') &= -i \begin{bmatrix} \langle \chi(x)\chi^\dagger(x')\rangle & \langle \chi(x)\chi(x')\rangle \\ \langle \chi^\dagger(x)\chi^\dagger(x')\rangle & \langle \chi^\dagger(x)\chi(x')\rangle \end{bmatrix}, \\ G^{<}(x, x') &= -i \begin{bmatrix} \langle \chi^\dagger(x')\chi(x)\rangle & \langle \chi(x')\chi(x)\rangle \\ \langle \chi^\dagger(x')\chi^\dagger(x)\rangle & \langle \chi(x')\chi^\dagger(x)\rangle \end{bmatrix}, \end{aligned} \tag{3.26}$$

or alternatively

$$G^{>}_{ab}(x, x') = -i\langle \Psi_a(x)\Psi^\dagger_b(x')\rangle, \quad G^{<}_{ab}(x, x') = -i\langle \Psi^\dagger_b(x')\Psi_a(x)\rangle, \tag{3.27}$$

where and hereafter $a, b = 1, 2$ denote the Nambu-Gor'kov indices. The expectation values in the expressions above are in the non-interacting thermal density matrix which corresponds to the quadratic part of the Lagrangian (Hamiltonian), i.e., the density matrix that describes free Bogoliubov quasiparticles in thermal equilibrium at inverse temperature β.

While the matrix elements $G^{\gtrless}_{ab}(x, x')$ can be obtained through the usual Bogoliubov transformation to the quasiparticle basis [11,6,28,8], we present in the Appendix an alternative derivation of these correlation functions directly from the spinor solutions of the homogeneous equations of motion. We find that the correlation functions in (3.26) in the continuum limit are given by

$$G^{\gtrless}_{ab}(x,x') = \int \frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} G^{\gtrless}_{ab}(k,t-t'), \tag{3.28}$$

where

$$\begin{aligned} G^{>}_{ab}(k,t-t') &= -i\Big[[1+n_B(\omega_k)]\mathcal{G}_{ab}(k)e^{-i\omega_k(t-t')} \\ &\qquad + n_B(\omega_k)\overline{\mathcal{G}}_{ab}(k)e^{i\omega_k(t-t')}\Big], \\ G^{<}_{ab}(k,t-t') &= -i\Big[n_B(\omega_k)\mathcal{G}_{ab}(k)e^{-i\omega_k(t-t')} \\ &\qquad + [1+n_B(\omega_k)]\overline{\mathcal{G}}_{ab}(k)e^{i\omega_k(t-t')}\Big]. \end{aligned} \tag{3.29}$$

In the above expressions, $k \equiv |\mathbf{k}|$, $n_B(\omega) = 1/(e^{\beta\omega}-1)$ is the Bose-Einstein distribution function,

$$\begin{aligned} \mathcal{G}(k) &= u_k^2 \begin{bmatrix} 1 & -r_k\phi_0/\phi_0^* \\ -r_k\phi_0^*/\phi_0 & r_k^2 \end{bmatrix}, \\ \overline{\mathcal{G}}(k) &= u_k^2 \begin{bmatrix} r_k^2 & -r_k\phi_0/\phi_0^* \\ -r_k\phi_0^*/\phi_0 & 1 \end{bmatrix}, \end{aligned} \tag{3.30}$$

where u_k, r_k are given by (A.5) in the Appendix, and ω_k is the energy of the free Bogoliubov quasiparticles (see the Appendix)

$$\omega_k = \left[\left(\frac{k^2}{2m} - \mu + 2g|\phi_0|^2\right)^2 - \left(g|\phi_0|^2\right)^2\right]^{1/2}. \tag{3.31}$$

Using the relation $1 + n_B(\omega) = e^{\beta\omega}\, n_B(\omega)$, one can easily verify the KMS condition [43]

$$G^{>}(k,t-i\beta-t') = G^{<}(k,t-t'). \tag{3.32}$$

Hence, the correlation functions for the fields $\chi^\dagger$, χ that will enter in the nonequilibrium perturbative expansion are completely determined by (3.28)-(3.31).

3.3 Feynman rules

From the generating functional of nonequilibrium Green's functions (3.23), it is clear that the effective interaction Lagrangian relevant for the nonequilibrium calculations is given by

$$\mathcal{L}_{\rm int}^{\rm eff}[\chi^{\dagger\pm}, \chi^{\pm}] = \mathcal{L}_{\rm int}[\chi^{\dagger+}, \chi^{+}] - \mathcal{L}_{\rm int}[\chi^{\dagger-}, \chi^{-}], \tag{3.33}$$

where the fields $\chi^{\dagger\pm}$ and $\chi^{\pm}$ are defined on the forward $(+)$ and backwards $(-)$ time branches respectively. Consequently, this generating functional leads to the following Feynman rules that define the perturbative expansion for calculations of nonequilibrium expectation values.

(i) There are *two* sets of interaction vertices defined by $\mathcal{L}_{\rm int}^{\rm eff}[\chi^{\dagger\pm}, \chi^{\pm}]$: those in which the fields are in the forward $(+)$ branch and those in which the fields are in the backward $(-)$ branch. There is a relative minus sign between these two types of vertices.
(ii) There are *four* sets of Green's functions given by (3.24) in terms of the normal and anomalous correlation functions in the form displayed in (3.26). These correlation functions are completely determined by (3.28)-(3.31).
(iii) The combinatoric factors are the same as in the equilibrium or imaginary-time (Matsubara) formulation.

4 Relaxation of condensate perturbations: An initial value problem

The equations of motion for the small amplitude condensate perturbation $\delta(x)$ induced by the external source $\eta(x)$ is obtained by implementing the *tadpole method* presented in Refs. [38,42]. This method begins by writing the original Bose fields in the forward $(+)$ and backward $(-)$ time branches as

$$\psi^{\pm}(x) = \phi_0 + \delta(x) + \chi^{\pm}(x), \tag{4.1}$$

where ϕ_0 is the homogeneous condensate in the absence of external source, $\delta(x)$ is the perturbation of the condensate induced by the external source which vanishes in the absence of external source and $\chi^{\pm}(x)$ are the noncondensate fields in the forward $(+)$ and backward $(-)$ time branches. The external source $\eta(x)$ are c-number fields and hence taken the same value in both forward and backward branches. The strategy to obtain the equations of motion for small amplitude condensate perturbation $\delta(x)$, $\delta^*(x)$ is to consider the *linear*, *cubic* and *quartic* terms in $\chi(x)$, $\chi^{\dagger}(x)$ in perturbation theory and impose the *tadpole condition*

$$\langle \chi^{\pm}(x) \rangle_{\eta} = \langle \chi^{\dagger\pm}(x) \rangle_{\eta} = 0 \tag{4.2}$$

order by order in the perturbative expansion, but, consistent with linear response, only keep contributions linear in $\delta(x)$, $\delta^*(x)$.

To illustrate this technique within the simplest setting and before embarking on the lengthy calculation at one-loop order, let us focus on obtaining the equations of motion for $\delta(x)$, $\delta^*(x)$ to lowest order in g by fulfilling the tadpole condition $\langle\chi^+(0)\rangle_\eta = 0$ to lowest order in the interaction. Since the lowest order interaction terms in $\mathcal{L}_{\rm int}[\chi^\dagger, \chi]$ are those *linear* in $\chi^\dagger$ and χ [see (2.13)], one obtains

$$
\begin{aligned}
\int d^4x \Bigg\{ &\langle\chi^+(0)\chi^{\dagger+}(x)\rangle \Bigg[\left(i\frac{\partial}{\partial t} + \frac{\nabla^2}{2m} + \mu - 2g|\phi_0|^2\right)\delta(x) - g\phi_0^2\delta^*(x) \\
&+ \phi_0\left(\mu - g|\phi_0|^2\right) + \eta(x)\Bigg] + \langle\chi^+(0)\chi^+(x)\rangle \Bigg[\Bigg(- i\frac{\partial}{\partial t} + \frac{\nabla^2}{2m} + \mu \\
&- 2g|\phi_0|^2\Bigg)\delta^*(x) - g\phi_0^{*2}\delta(x) + \phi_0^*\left(\mu - g|\phi_0|^2\right) + \eta^*(x)\Bigg] \\
&- \langle\chi^+(0)\chi^{\dagger-}(x)\rangle \Bigg[\left(i\frac{\partial}{\partial t} + \frac{\nabla^2}{2m} + \mu - 2g|\phi_0|^2\right)\delta(x) - g\phi_0^2\delta^*(x) \\
&+ \phi_0\left(\mu - g|\phi_0|^2\right) + \eta(x)\Bigg] - \langle\chi^+(0)\chi^-(x)\rangle \Bigg[\Bigg(- i\frac{\partial}{\partial t} + \frac{\nabla^2}{2m} + \mu \\
&- 2g|\phi_0|^2\Bigg)\delta^*(x) - g\phi_0^{*2}\delta(x) + \phi_0^*\left(\mu - g|\phi_0|^2\right) + \eta^*(x)\Bigg]\Bigg\} = 0,
\end{aligned}
\tag{4.3}
$$

which lead to the lowest order equations of motion for $\delta(\mathbf{x}, t)$ and $\delta^*(\mathbf{x}, t)$

$$
\begin{aligned}
&\left(i\frac{\partial}{\partial t} + \frac{\nabla^2}{2m} + \mu - 2g|\phi_0|^2\right)\delta(\mathbf{x}, t) - g\phi_0^2\delta^*(\mathbf{x}, t) \\
&\qquad\qquad + \phi_0\left(\mu - g|\phi_0|^2\right) + \eta(\mathbf{x}, t) = 0, \\
&\left(-i\frac{\partial}{\partial t} + \frac{\nabla^2}{2m} + \mu - 2g|\phi_0|^2\right)\delta^*(\mathbf{x}, t) - g\phi_0^{*2}\delta(\mathbf{x}, t) \\
&\qquad\qquad + \phi_0^*\left(\mu - g|\phi_0|^2\right) + \eta^*(\mathbf{x}, t) = 0.
\end{aligned}
\tag{4.4}
$$

The equations of motion obtained from the tadpole condition $\langle\chi^-(0)\rangle_\eta = 0$ are the same as those given in the above expressions. This set of equations is recognized as the Gross-Pitaevskii equations [13,6,7] for the condensate wave function linearized for a small amplitude perturbation around a homogeneous condensate ϕ_0.

In order to generalize the features gleaned from the lowest order equations of motion to higher orders in the loop expansion, we rewrite the above equations in the following illuminating form

$$\left(i\frac{\partial}{\partial t}+\frac{\nabla^2}{2m}+\mu\right)\delta(\mathbf{x},t)-\Sigma_{11}^{(0)}\delta(\mathbf{x},t)-\Sigma_{12}^{(0)}\delta^*(\mathbf{x},t)+\mathcal{T}[\phi_0,\phi_0^*]=-\eta(\mathbf{x},t),$$
$$\left(-i\frac{\partial}{\partial t}+\frac{\nabla^2}{2m}+\mu\right)\delta^*(\mathbf{x},t)-\Sigma_{22}^{(0)}\delta^*(\mathbf{x},t)-\Sigma_{21}^{(0)}\delta(\mathbf{x},t)+\mathcal{T}^*[\phi_0,\phi_0^*]=-\eta^*(\mathbf{x},t),$$
$$(4.5)$$

where the normal and anomalous self-energies are given by

$$\Sigma_{11}^{(0)}=\Sigma_{22}^{(0)}=2g|\phi_0|^2,\quad \Sigma_{12}^{(0)}=\Sigma_{21}^{(0)*}=g\phi_0^2, \tag{4.6}$$

respectively, and the *tadpole* $\mathcal{T}[\phi_0,\phi_0^*]$ denotes terms independent of the condensate perturbation δ and δ^*

$$\mathcal{T}[\phi_0,\phi_0^*]=\phi_0\left(\mu-g|\phi_0|^2\right). \tag{4.7}$$

Before proceeding further, we note that whereas this lowest order calculation is fairly straightforward, the equations obtained above feature the following important aspects that will be shared also by the higher order calculations.

(i) The last two lines in the tadpole condition (4.3), which are obtained by replacing $\chi^{\dagger+}\to\chi^{\dagger-}$ and $\chi^+\to\chi^-$ in the correlation functions, lead to the same equations of motion as those obtained from the first two lines. This is a generic feature to all orders in the perturbative expansion [38] and is a consequence of the fact that the coefficients of the correlation functions are local functions of t which must be the same in both branches [38,42].

(ii) If the homogeneous condensate $|\phi_0|^2\neq\mu/g$ then the tadpole $\mathcal{T}[\phi_0,\phi_0^*]$ in (4.4) acts as a driving force for δ, which will therefore be nonvanishing. Hence, the requirement that the departure from equilibrium δ vanishes in the absence of external source, i.e., when $\eta\equiv 0$ at all times, implies that the homogeneous condensate must fulfill the lowest order *equilibrium condition*

$$\mu-g|\phi_0|^2=0. \tag{4.8}$$

In other words if and only if the homogeneous condensate fulfills the equilibrium condition (4.8), the perturbation δ will vanish when the external source η vanishes at all times.

(iii) Setting δ, $\eta=0$ in the interaction Lagrangian (2.13) the tadpole condition $\langle\chi^+(0)\rangle_\eta=0$ leads to the equation for the homogeneous condensate

$$\mathcal{T}[\phi_0,\phi_0^*]=\phi_0\left(\mu-g|\phi_0|^2\right)=0, \tag{4.9}$$

which determines the equilibrium value of the homogeneous condensate.

Now consider adding an external space-time independent source term η, which leads to a space-time independent shift of the condensate $\phi_0 \to \phi_0 + \delta$. With the homogeneous condensate now being $\phi_0 + \delta$ the corresponding equation of motion is therefore given by

$$\mathcal{T}[\phi_0 + \delta, \phi_0^* + \delta^*] = -\eta. \tag{4.10}$$

Expanding to linear order in δ and δ^*, this equation becomes

$$\mathcal{T}[\phi_0, \phi_0^*] + \frac{\partial \mathcal{T}[\phi_0, \phi_0^*]}{\partial \phi_0}\, \delta + \frac{\partial \mathcal{T}[\phi_0, \phi_0^*]}{\partial \phi_0^*}\, \delta^* = -\eta, \tag{4.11}$$

which, upon comparing to the first equation of motion in (4.5) for a space-time constant perturbation, leads to the identification

$$\frac{\partial \mathcal{T}[\phi_0, \phi_0^*]}{\partial \phi_0} = \mu - \Sigma_{11}^{(0)}, \quad \frac{\partial \mathcal{T}[\phi_0, \phi_0^*]}{\partial \phi_0^*} = -\Sigma_{12}^{(0)}. \tag{4.12}$$

Furthermore, if the condition (4.9) is fulfilled, then using the expressions for the self-energies (4.6) we obtain

$$\mu = \Sigma_{11}^{(0)} - \frac{\phi_0^*}{\phi_0} \Sigma_{12}^{(0)}. \tag{4.13}$$

Assuming that the condensate ϕ_0 is real, which can always be made by a gauge transformation (2.14), the above expression is recognized as the Hugenholtz-Pines relation [22].

In general, for an arbitrary phase of the condensate, the lowest order expression for the Hugenholtz-Pines theorem (4.13) accounts for the fact that $\Sigma_{12}^{(0)}$ must be complex by gauge invariance. Indeed, $\Sigma_{11}^{(0)}$ must be invariant under the gauge transformation (2.14) since it multiplies δ, whereas $\Sigma_{12}^{(0)}$ must transform as an object with charge $+2$ since it multiplies δ^* and the full equation of motion for δ must transform as δ itself. Similarly, $\Sigma_{21}^{(0)}$ must transform as an object with charge -2 under the gauge transformation (2.14). This is an important point that will be taken up again in higher order calculations and will be the basis for the alternative derivation of the Hugenholtz-Pines theorem to all orders in perturbation theory (see Sec. 5).

While these features emerge as rather trivial relationships that are gleaned directly from the lowest order (classical) equations of motion, they will manifest themselves to *all orders* in the perturbation expansion in the tadpole method. The much less trivial statement that follows to all orders in the perturbative expansion implemented via the tadpole method and that will be confirmed by an explicit calculation below, is that this method leads to a *gapless* approximation in the classification of Hohenberg and Martin [21].

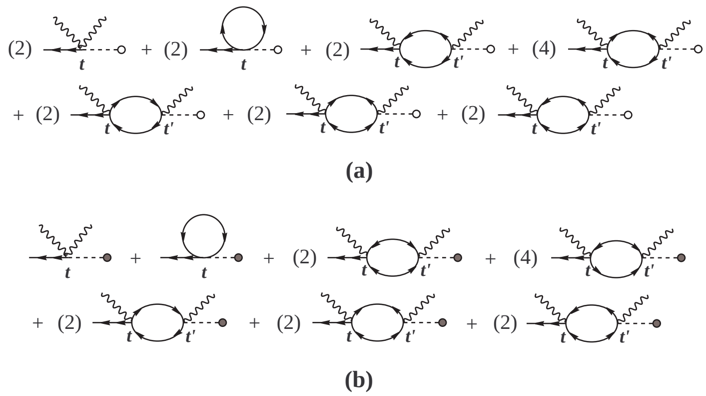

Fig. 2. Feynman diagrams contributing to the self-energies (a) $\Sigma_{11}^{++}(x-x')$ and (b) $\Sigma_{12}^{++}(x-x')$ up to one-loop order with the combinatoric factors denoted in parenthesis and the labels representing the time branches suppressed. A solid line with arrows denotes the noncondensate particle, a wiggly line denotes the condensate ϕ_0 or ϕ_0^*, and a dashed line with open (closed) circle denotes the condensate perturbation δ (δ^*).

Furthermore, the lowest order equilibrium condition (4.8) relates the equilibrium condensate at this order to the chemical potential by $\mu = g|\phi_0|^2$, which is the lowest order Hugenholtz-Pines relation (4.13) and hence leads to a gapless spectrum for free Bogoliubov quasiparticle excitations in equilibrium condensate

$$\omega_k = \left[\left(\frac{k^2}{2m} + g|\phi_0|^2\right)^2 - \left(g|\phi_0|^2\right)^2\right]^{1/2}, \tag{4.14}$$

as can be easily verified from (3.31).

4.2 Equations of motion at one-loop order

The simple example above highlights the tadpole method to obtain the equations of motion for $\delta(x)$ and $\delta^*(x)$, which we now pursue to one-loop order. The strategy is the same, but at one-loop order we now face many Feynman diagrams by treating the effective Lagrangian $\mathcal{L}_{\rm int}^{\rm eff}[\chi^{\dagger\pm}, \chi^{\pm}]$ in perturbation theory with the Feynman rules described above. Just as in the simpler lowest order example worked out above, there are *two* types of contributions to the equations of motion: (i) contributions that are proportional to δ, δ^*, these give the corresponding self-energies, and (ii) contributions that are *independent* of δ, δ^*. The latter are the higher order generalization of the tadpole $\mathcal{T}[\phi_0, \phi_0^*]$ and its complex conjugate in (4.5).

Using the Feynman rules described in the previous section, the tadpole condition $\langle \chi^+(0) \rangle_\eta = 0$ leads to the following expression

$$
\begin{aligned}
&\int d^4x \Bigg\{ \langle \chi^+(0)\chi^{\dagger +}(x)\rangle \Bigg[\left(i\frac{\partial}{\partial t} + \frac{\nabla^2}{2m} + \mu \right) \delta(x) - \int d^4x' \Big[\Sigma_{11}(x-x')\delta(x') \\
&\quad + \Sigma_{12}(x-x')\delta^*(x') \Big] + \mathcal{T}[\phi_0, \phi_0^*] + \eta(x) \Bigg] + \langle \chi^+(0)\chi^+(x)\rangle \Bigg[\Bigg(- i\frac{\partial}{\partial t} \\
&\quad + \frac{\nabla^2}{2m} + \mu \Bigg) \delta^*(x) - \int d^4x' \Big[\Sigma_{22}(x-x')\delta^*(x') + \Sigma_{21}(x-x')\delta(x') \Big] \\
&\quad + \mathcal{T}^*[\phi_0, \phi_0^*] + \eta^*(x) \Bigg] - \Big[\chi^{\dagger +}(x) \to \chi^{\dagger -}(x), \chi^+(x) \to \chi^-(x) \Big] \Bigg\} = 0,
\end{aligned}
\tag{4.15}
$$

where $\Sigma_{ab}(x-x') = \Sigma_{ab}^{++}(x-x') + \Sigma_{ab}^{+-}(x-x')$ are the self-energies and $\mathcal{T}[\phi_0, \phi_0^*]$ is the tadpole. The diagrams for the self-energies Σ_{11}^{++} and Σ_{12}^{++} up to one-loop order are depicted in Fig. 2 with the labels representing the time branches suppressed for simplicity, those for Σ_{11}^{+-} and Σ_{12}^{+-} can be obtained in an analogous manner. The diagrams for the tadpole $\mathcal{T}[\phi_0, \phi_0^*]$ are depicted in Fig. 3.

Introducing the space Fourier transforms for $\delta(x)$, $\eta(x)$ and $\Sigma_{ab}(x-x')$ as

$$
\delta(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^3}\, \delta_{\mathbf{k}}(t)\, e^{i\mathbf{k}\cdot\mathbf{x}}, \text{ etc.,} \tag{4.16}
$$

we find the final form of the equations of motion in momentum space to be given by

$$
\begin{aligned}
&\left(i\frac{d}{dt} - \frac{k^2}{2m} + \mu \right) \delta_{\mathbf{k}}(t) - \int_{-\infty}^{+\infty} dt' \Big[\Sigma_{11}(\mathbf{k}, t-t')\, \delta_{\mathbf{k}}(t') + \Sigma_{12}(\mathbf{k}, t-t') \\
&\qquad \times \delta_{-\mathbf{k}}^*(t') \Big] + \mathcal{T}[\phi_0, \phi_0^*]\, \delta^{(3)}(\mathbf{k}) + \eta_{\mathbf{k}}(t) = 0, \\
&\left(-i\frac{d}{dt} - \frac{k^2}{2m} + \mu \right) \delta_{-\mathbf{k}}^*(t) - \int_{-\infty}^{+\infty} dt' \Big[\Sigma_{22}(\mathbf{k}, t-t')\, \delta_{-\mathbf{k}}^*(t') + \Sigma_{21}(\mathbf{k}, t-t') \\
&\qquad \times \delta_{\mathbf{k}}(t') \Big] + \mathcal{T}^*[\phi_0, \phi_0^*]\, \delta^{(3)}(\mathbf{k}) + \eta_{-\mathbf{k}}^*(t) = 0.
\end{aligned}
\tag{4.17}
$$

The equations of motion obtained from the tadpole condition $\langle \chi^-(0)\rangle_\eta = 0$ are the same as those given in the above expressions. While the above expression has been obtained at one-loop order, it is straightforward to conclude after a simple diagrammatic analysis that the structure of the equations of motion obtained above is *general and valid to all orders*.

Figure 4, which displays the diagrams for the normal self-energy Σ_{11}, clearly shows that the self-energies have a local, instantaneous contribution (the first two diagrams in the first bracket), which are known as the Hartree-Fock-Bogoliubov (HFB) contributions, and a nonlocal, retarded contribution (the rest of the diagrams) that will lead to absorptive parts. A simple calculation

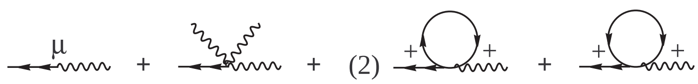

Fig. 3. Feynman diagrams contributing to the tadpole $\mathcal{T}[\phi_0, \phi_0^*]$ up to one-loop order.

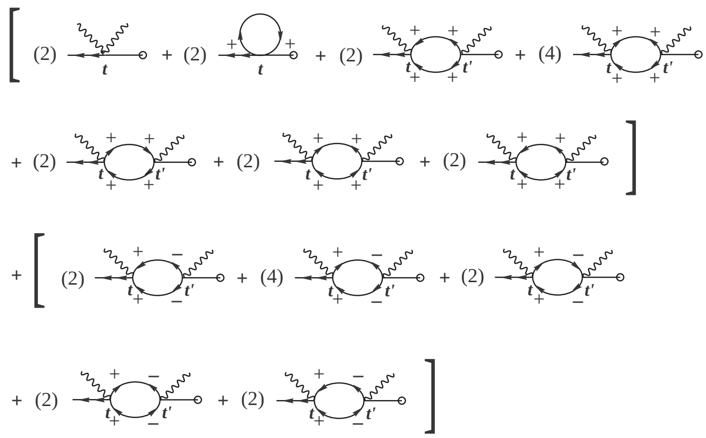

Fig. 4. Feynman diagrams contributing to the normal self-energy $\Sigma_{11}(x-x')$. Those for the anomalous self-energy $\Sigma_{12}(x-x')$ can be obtained from the diagrams in Fig. 2(b) by attaching the labels $\pm$ in the same manner as those depicted in this figure.

with the real-time nonequilibrium Green's functions obtained in the previous section reveals that

$$\begin{aligned}\Sigma_{11}(\mathbf{k}, t-t') &= \Sigma_{11}^{\mathrm{HFB}}\,\delta(t-t') + \Sigma_{11}^{R}(k, t-t')\,\Theta(t-t'),\\ \Sigma_{12}(\mathbf{k}, t-t') &= \Sigma_{12}^{\mathrm{HFB}}\,\delta(t-t') + \Sigma_{12}^{R}(k, t-t')\,\Theta(t-t').\end{aligned} \tag{4.18}$$

From the explicit expressions for the self-energies or by taking complex conjugation of the equation of motion for $\delta_{\mathbf{k}}(t)$, we find the properties

$$\begin{aligned}\Sigma_{21}(\mathbf{k}, t-t') &= \Sigma_{12}^{*}(-\mathbf{k}, t-t'),\\ \Sigma_{22}(\mathbf{k}, t-t') &= \Sigma_{11}^{*}(-\mathbf{k}, t-t').\end{aligned} \tag{4.19}$$

Furthermore, rotational and parity invariance imply that the self-energies are only functions of k (as can be explicitly confirmed at one-loop order).

The HFB instantaneous parts $\Sigma_{11}^{\mathrm{HFB}}$ and $\Sigma_{12}^{\mathrm{HFB}}$ including the lowest order contributions (4.6) are given by

$$\begin{aligned}\Sigma_{11}^{\mathrm{HFB}} &= 2g(n_0 + \widetilde{n}), \quad \Sigma_{12}^{\mathrm{HFB}} = g(\phi_0^2 + \widetilde{m}),\\ n_0 &= |\phi_0|^2, \quad \widetilde{n} = \langle \chi^\dagger \chi \rangle, \quad \widetilde{m} = \langle \chi\chi \rangle,\end{aligned} \tag{4.20}$$

where n_0 is the density of the condensate particles, $\widetilde{n}$ and $\widetilde{m}$ are the one-loop normal and anomalous densities of the noncondensate particles, respectively.

Using the expressions for the real-time correlation functions given by (3.26) with (3.29) and (3.30) in the coincidence limit (i.e., $t' \to t$) and the properties of the Bogoliubov coefficients given by (A.5) in the Appendix, we find

$$
\begin{aligned}
\widetilde{n} &= i \int \frac{d^3q}{(2\pi)^3}\, G^{<}_{11}(q,0) \\
&= \int \frac{d^3q}{(2\pi)^3} u_q^2 \left[n_B(\omega_q) + r_q^2 [1 + n_B(\omega_q)] \right], \\
\widetilde{m} &= i \int \frac{d^3q}{(2\pi)^3} G^{>}_{12}(q,0) \\
&= -\frac{g\phi_0^2}{2} \int \frac{d^3q}{(2\pi)^3 \omega_q} [1 + 2n_B(\omega_q)],
\end{aligned}
\tag{4.21}
$$

where in obtaining the final expression of $\widetilde{m}$ use has been made of (A.5) in the Appendix.

As argued above, the normal self-energies Σ_{11} and Σ_{22} must be invariant under the gauge transformation (2.14), while the anomalous ones Σ_{12} and Σ_{21} must transform as ϕ_0^2 and ϕ_0^{*2}, respectively. Thus using the property (4.19) it is proves convenient to write

$$
\begin{aligned}
\Sigma_{11}(k, t-t') = \Sigma^*_{22}(k, t-t') = \Sigma_D(k, t-t'), \\
\Sigma_{12}(k, t-t') = \Sigma^*_{21}(k, t-t') = (\phi_0/\phi_0^*)\, \Sigma_O(k, t-t'),
\end{aligned}
\tag{4.22}
$$

where both Σ_D andΣ_O are *invariant* under the gauge transformation. While rewriting the self-energies in this manner may seem a redundant exercise, the main point is to highlight and make explicit their transformation laws under the gauge transformation. This is an important aspect that needs to be addressed carefully in order to extract the Ward identities, an *exact* result of the underlying gauge symmetry to be explored below.

The gauge invariant HFB instantaneous parts $\Sigma_D^{\rm HFB}$ and $\Sigma_O^{\rm HFB}$ can be obtained straightforwardly from (4.20), whereas the gauge invariant nonlocal, retarded parts Σ_D^R and Σ_O^R can be written in terms of their spectral representation as

$$
\begin{aligned}
\Sigma^R_D(k, t-t') &= \int_{-\infty}^{+\infty} d\omega \Big\{ i\, [\overline{S}_B(k,\omega) + \overline{S}_L(k,\omega)] \cos\omega(t-t') \\
&\qquad + [\overline{A}_B(k,\omega) + \overline{A}_L(k,\omega)] \sin\omega(t-t') \Big\}, \\
\Sigma^R_O(k, t-t') &= \int_{-\infty}^{+\infty} d\omega\, [\widehat{A}_B(k,\omega) + \widehat{A}_L(k,\omega)] \sin\omega(t-t').
\end{aligned}
\tag{4.23}
$$

The symmetric $[S(k,\omega)]$ and antisymmetric $[A(k,\omega)]$ spectral functions are even and odd functions of ω, respectively, and found to be given by (see the Appendix for the explicit expressions of u_q and r_q)

$$
\begin{aligned}
\overline{S}_B(k,\omega) = & -g^2 n_0 \int \frac{d^3q}{(2\pi)^3}\, u_q^2 u_p^2 \left[r_p^2(2-4r_q+r_q^2)-(1-4r_q+2r_q^2)\right] \\
& \times [1+n_B(\omega_q)+n_B(\omega_p)][\delta(\omega-\omega_q-\omega_p)+\delta(\omega+\omega_q+\omega_p)], \\
\overline{S}_L(k,\omega) = & -g^2 n_0 \int \frac{d^3q}{(2\pi)^3}\, u_q^2 u_p^2 \left[r_p^2(1-4r_q+2r_q^2)-(2-4r_q+r_q^2)\right] \\
& \times [n_B(\omega_q)-n_B(\omega_p)][\delta(\omega-\omega_q+\omega_p)+\delta(\omega+\omega_q-\omega_p)], \\
\overline{A}_B(k,\omega) = & -g^2 n_0 \int \frac{d^3q}{(2\pi)^3}\, u_q^2 u_p^2 \left[r_p^2(2-4r_q+r_q^2)+(1-4r_q+2r_q^2)+4r_q r_p\right] \\
& \times [1+n_B(\omega_q)+n_B(\omega_p)][\delta(\omega-\omega_q-\omega_p)-\delta(\omega+\omega_q+\omega_p)], \\
\overline{A}_L(k,\omega) = & -g^2 n_0 \int \frac{d^3q}{(2\pi)^3}\, u_q^2 u_p^2 \left[r_p^2(1-4r_q+2r_q^2)+(2-4r_q+r_q^2)+4r_p r_q\right] \\
& \times [n_B(\omega_q)-n_B(\omega_p)][\delta(\omega-\omega_q+\omega_p)-\delta(\omega+\omega_q-\omega_p)], \\
\widehat{A}_B(k,\omega) = & -2g^2 n_0 \int \frac{d^3q}{(2\pi)^3}\, u_q^2 u_p^2 \left[2r_q^2-2r_p(1+r_q^2)+3r_p r_q\right] \\
& \times [1+n_B(\omega_q)+n_B(\omega_p)][\delta(\omega-\omega_q-\omega_p)-\delta(\omega+\omega_q+\omega_p)], \\
\widehat{A}_L(k,\omega) = & -2g^2 n_0 \int \frac{d^3q}{(2\pi)^3}\, u_q^2 u_p^2 \left[1+r_p^2 r_q^2-2r_p(1+r_q^2)+3r_p r_q\right] \\
& \times [n_B(\omega_q)-n_B(\omega_p)][\delta(\omega-\omega_q+\omega_p)-\delta(\omega+\omega_q-\omega_p)], \quad (4.24)
\end{aligned}
$$

where $\mathbf{p} = \mathbf{k} + \mathbf{q}$. The subscript B denotes the Beliaev damping processes in which one quasiparticle decays into two and the inverse process [16,17], and L denotes the Landau damping processes in which the quasiparticle scatters off another quasiparticle in the noncondensate [18–20]. The Landau damping contributions to the nonlocal, retarded parts of the self-energies arise solely from the thermally excited quasiparticles and hence vanish in the zero temperature limit. A careful comparison of the spectral representation of the self-energies reveals that they coincide with those obtained by Shi and Griffin [8].

The tadpole term up to one-loop order is given by

$$
\mathcal{T}[\phi_0,\phi_0^*] = \phi_0 \left[\mu - g\left(n_0 + 2\widetilde{n} + \frac{\phi_0^*}{\phi_0}\widetilde{m}\right)\right], \qquad (4.25)
$$

with $\widetilde{n}$ and $\widetilde{m}$ given by (4.21). Setting the external source $\eta = 0$, the equilibrium condition for the homogeneous condensate $\mathcal{T}[\phi_0,\phi_0^*] = 0$ leads to

$$
\phi_0 \left[\mu - g\left(n_0 + 2\widetilde{n} + \frac{\phi_0^*}{\phi_0}\widetilde{m}\right)\right] = 0. \qquad (4.26)
$$

In equilibrium and below the critical temperature, the condensate $\phi_0 \neq 0$, therefore (4.26) provides one relationship between the equilibrium condensate and the chemical potential. The other relationship is determined by fixing the total number of particles, i.e.,

$$n = n_0 + \tilde{n}, \tag{4.27}$$

where n is the total density of the particles. Thus, in equilibrium the two conditions (4.26) and (4.27) completely determine the chemical potential and the equilibrium condensate consistently to one-loop order.

4.3 Nonequilibrium dynamics as an initial value problem

In an experimental situation the dynamical evolution of the small amplitude condensate perturbation is studied by preparing a Bose-condensed gas slightly perturbed away from equilibrium by adiabatically coupling to some external source in the infinite past. Once the source is switched-off at time $t = 0$ the perturbed condensate relaxes towards equilibrium and the relaxation dynamics is studied. As discussed above, this experimental situation can be realized within the real-time formulation described here by taking the spatial Fourier transform of external source to be of the form

$$\eta_{\mathbf{k}}(t) = \eta_{\mathbf{k}}\, e^{\epsilon t}\, \Theta(-t), \quad \epsilon \to 0^+. \tag{4.28}$$

The ϵ-term serves to switch on the source adiabatically from $t = -\infty$ so as not to disturb the system too far from equilibrium in the process. If at $t = -\infty$ the system was in an equilibrium state, then the condition of equilibrium (4.26) ensures that for $t < 0$ there is a solution of the equations of motion (4.17) of the form

$$\delta_{\mathbf{k}}(t) = \delta_{\mathbf{k}}(0)\, e^{\epsilon t} \quad \text{for} \;\; t < 0, \tag{4.29}$$

where $\delta_{\mathbf{k}}(0)$ is related to $\eta_{\mathbf{k}}$ through the equations of motion for $t < 0$. The advantage of the adiabatic switching-on of the external source is that the time derivative of the solution (4.29) satisfies $\dot{\delta}_{\mathbf{k}}(t < 0) \to 0$ as $\epsilon \to 0^+$.

Introducing auxiliary quantities $\Pi^R_{ab}(k, t - t')$ defined as

$$\Sigma^R_{ab}(k, t - t') = \frac{\partial}{\partial t'} \Pi^R_{ab}(k, t - t'), \tag{4.30}$$

then upon using integration by parts, neglecting terms that vanish in the adiabatic limit $\epsilon \to 0^+$ and assuming that ϕ_0 is the equilibrium condensate and hence $\mathcal{T}[\phi_0, \phi_0^*] = 0$, the equations of motion (4.17) for $t > 0$ are given by

$$
\begin{aligned}
&\left[i\frac{d}{dt} - \frac{k^2}{2m} + \mu - \Sigma_{11}^{\rm HFB} - \Pi_{11}^R(k,0)\right]\delta_{\mathbf{k}}(t) - [\Sigma_{12}^{\rm HFB} - \Pi_{12}^R(k,0)]\delta^*_{-\mathbf{k}}(t) \\
&\qquad + \int_0^t dt' \Big[\Pi_{11}^R(k,t-t')\dot{\delta}_{\mathbf{k}}(t') + \Pi_{12}^R(k,t-t')\dot{\delta}^*_{-\mathbf{k}}(t')\Big] = 0, \\
&\left[-i\frac{d}{dt} - \frac{k^2}{2m} + \mu - \Sigma_{22}^{\rm HFB} - \Pi_{22}^R(k,0)\right]\delta^*_{-\mathbf{k}}(t) - [\Sigma_{21}^{\rm HFB} + \Pi_{21}^R(k,0)]\delta_{\mathbf{k}}(t) \\
&\qquad + \int_0^t dt' \Big[\Pi_{22}^R(k,t-t')\dot{\delta}^*_{-\mathbf{k}}(t') + \Pi_{21}^R(k,t-t')\dot{\delta}_{\mathbf{k}}(t')\Big] = 0.
\end{aligned}
\tag{4.31}
$$

The above coupled equations of motion for the condensate perturbation are now in the form of an *initial value problem* with initial conditions specified at $t = 0$ and can be solved by Laplace transform. Introducing a two-component Nambu-Gor'kov spinor and its Laplace transform

$$
\Delta_{\mathbf{k}}(t) = \begin{bmatrix} \delta_{\mathbf{k}}(t) \\ \delta^*_{-\mathbf{k}}(t) \end{bmatrix}, \quad \widetilde{\Delta}_{\mathbf{k}}(s) = \begin{bmatrix} \widetilde{\delta}_{\mathbf{k}}(s) \\ \widetilde{\delta}^*_{-\mathbf{k}}(s) \end{bmatrix}, \tag{4.32}
$$

where s is the complex Laplace variable with $\mathrm{Re}\, s > 0$, one can write the Laplace transformed equations of motion in a compact matrix form as

$$
\widetilde{G}^{-1}(k,s)\,\widetilde{\Delta}_{\mathbf{k}}(s) = \frac{1}{s}\Big[\widetilde{G}^{-1}(k,s) - \widetilde{G}^{-1}(k,0)\Big]\Delta_{\mathbf{k}}(0). \tag{4.33}
$$

In the above equation, $\widetilde{G}^{-1}(k,s)$ is the inverse Green's function (matrix) up to one-loop order expressed in terms of the Laplace variable s

$$
\widetilde{G}^{-1}(k,s) = \begin{bmatrix} is - k^2/2m + \mu - \widetilde{\Sigma}_D(k,s) & -(\phi_0/\phi_0^*)\widetilde{\Sigma}_O(k,s) \\ -(\phi_0^*/\phi_0)\widetilde{\Sigma}_O(k,s) & -is - k^2/2m + \mu - \widetilde{\Sigma}_D(k,-s) \end{bmatrix}, \tag{4.34}
$$

with

$$
\widetilde{\Sigma}_D(k,s) = \Sigma_D^{\rm HFB} + \widetilde{\Sigma}_D^R(k,s), \quad \widetilde{\Sigma}_O(k,s) = \Sigma_O^{\rm HFB} + \widetilde{\Sigma}_O^R(k,s), \tag{4.35}
$$

where $\widetilde{\Sigma}_D^R(k,s)$ and $\widetilde{\Sigma}_O^R(k,s)$ are the Laplace transforms of $\Sigma_D^R(k,t-t')$ and $\Sigma_O^R(k,t-t')$, respectively,

$$
\begin{aligned}
\widetilde{\Sigma}_D^R(k,s) &= \int_{-\infty}^{+\infty} \frac{dk_0}{k_0 - is}\left[\overline{S}_B(k,k_0) + \overline{S}_L(k,k_0) + \overline{A}_B(k,k_0) + \overline{A}_L(k,k_0)\right], \\
\widetilde{\Sigma}_O^R(k,s) &= \int_{-\infty}^{+\infty} \frac{dk_0}{k_0 - is}\Big[\widehat{A}_B(k,k_0) + \widehat{A}_L(k,k_0)\Big].
\end{aligned}
\tag{4.36}
$$

In obtaining (4.34), we have made use of the properties

$$\widetilde{\Sigma}_{22}(k,s) = \widetilde{\Sigma}_{11}(k,-s), \quad \widetilde{\Sigma}_{21}(k,s) = (\phi_0^*/\phi_0)^2\, \widetilde{\Sigma}_{12}(k,s), \tag{4.37}$$

which are a result of (4.19) and (4.23). The solution of (4.34) reads

$$\widetilde{\Delta}_{\mathbf{k}}(s) = \frac{1}{s}\Big[1 - \widetilde{G}(k,s)\, \widetilde{G}^{-1}(k,0)\Big]\Delta_{\mathbf{k}}(0), \tag{4.38}$$

where

$$\begin{aligned}\widetilde{G}(k,s) = & \frac{1}{\widetilde{D}(k,s)} \\ & \times \begin{bmatrix} is + k^2/2m - \mu + \widetilde{\Sigma}_D(k,-s) & -(\phi_0/\phi_0^*)\widetilde{\Sigma}_O(k,s) \\ -(\phi_0^*/\phi_0)\widetilde{\Sigma}_O(k,s) & -is + k^2/2m - \mu + \widetilde{\Sigma}_D(k,s) \end{bmatrix},\end{aligned} \tag{4.39}$$

with the denominator $\widetilde{D}(k,s)$ given by

$$\begin{aligned}\widetilde{D}(k,s) = & \Big[is - k^2/2m + \mu - \widetilde{\Sigma}_D(k,s)\Big]\Big[is + k^2/2m - \mu + \widetilde{\Sigma}_D(k,-s)\Big] \\ & + \widetilde{\Sigma}_O^2(k,s).\end{aligned} \tag{4.40}$$

The real-time evolution of the condensate perturbation $\Delta_{\mathbf{k}}(t)$ with an initial value $\Delta_{\mathbf{k}}(0)$ is now obtained from the inverse Laplace transform

$$\Delta_{\mathbf{k}}(t) = \int_{\mathcal{B}} \frac{ds}{2\pi i}\, e^{st}\, \widetilde{\Delta}_{\mathbf{k}}(s), \tag{4.41}$$

where the Bromwich contour $\mathcal{B}$ runs parallel to the imaginary axis in the complex s plane to the right of all the singularities (poles and cuts) of $\widetilde{\Delta}_{\mathbf{k}}(s)$ [39]. We note that there is *no* isolated pole in $\widetilde{\Delta}_{\mathbf{k}}(s)$ at $s = 0$ since the residue vanishes.

The Green's functions $\widetilde{G}(k,s)$ obtained above (albeit expressed in terms of the Laplace variable s) agree with those obtained via the Dyson's equations [16,8]. However, as discussed in detail in Refs. [16,8] that in equilibrium the Beliaev self-energies possess infrared divergences in the limit $k, s \to 0$ and hence care must be taken in analyzing the analytic properties of $\widetilde{\Delta}_{\mathbf{k}}(s)$.

At zero temperature these infrared divergences are logarithmic, while at finite temperatures, the Bose-Einstein enhancement factor enhances the infrared divergences to a power law. At zero temperature the infrared divergences that appear in the Beliaev self-energies were first analyzed by Gavoret and

Nozières [24], who showed that they cancel out exactly in the final expressions for physical quantities to all orders in perturbation theory. Nepomnyashchy and Nepomnyashchy [25] as well as Popov and Seredniakov [26] have analyzed the infrared divergences at finite temperature and also concluded that a resummation of the most infrared divergent diagrams leads to a finite result.

Having set up the description of the nonequilibrium dynamics of perturbations away from the equilibrium condensate as an initial value problem, we will defer the analysis of the real-time evolution and the infrared divergences in the self-energies to a forthcoming article.

5 Ward identity and Hugenholtz-Pines theorem

5.1 General results

A straightforward diagrammatic analysis with the Feynman rules described above reveals that the generic structure of the equations of motion obtained via the tadpole method remains the same to all orders in perturbation theory. When combined with the transformation properties of ϕ_0, δ and χ under the gauge transformation (2.14) this general form of the equations of motion allows to derive to all orders in perturbation theory the Ward identity for the tadpole $\mathcal{T}[\phi_0, \phi_0^*]$. As described below, this identity is a consequence of the underlying gauge symmetry and in equilibrium leads to an alternative derivation of the Hugenholtz-Pines theorem.

First, consider the case in which δ, $\eta = 0$ the tadpole condition $\langle \chi^+(0) \rangle_\eta = 0$ leads to

$$\mathcal{T}[\phi_0, \phi_0^*] = 0, \tag{5.1}$$

which is the equilibrium condition for the homogeneous condensate. For a space-time independent shift of the condensate $\phi_0 \to \phi_0 + \delta$ induced by a space-time independent source η, the tadpole condition now leads to

$$\mathcal{T}[\phi_0 + \delta, \phi_0^* + \delta^*] = -\eta, \tag{5.2}$$

which expanding to linear order in δ and δ^* becomes

$$\mathcal{T}[\phi_0, \phi_0^*] + \frac{\partial \mathcal{T}[\phi_0, \phi_0^*]}{\partial \phi_0} \delta + \frac{\partial \mathcal{T}[\phi_0, \phi_0^*]}{\partial \phi_0^*} \delta^* = -\eta. \tag{5.3}$$

We now compare this equation with the first equation of motion in (4.17), which has the same generic structure as the full equation of motion obtained to all orders in perturbation theory, for space-time independent δ and δ^*. We recognize

$$\frac{\partial \mathcal{T}[\phi_0,\phi_0^*]}{\partial \phi_0} = \mu - \Sigma_{11}(k=0,\omega=0),$$
$$\frac{\partial \mathcal{T}[\phi_0,\phi_0^*]}{\partial \phi_0^*} = -\Sigma_{12}(k=0,\omega=0), \tag{5.4}$$

to *all orders in perturbation theory.* This is obviously an *all-order* generalization of the lowest order result (4.12).

The second important ingredient and which stems from the equations of motion (4.17) is that under a gauge transformation (2.14) the tadpole $\mathcal{T}[\phi_0,\phi_0^*]$ transforms just as δ, ϕ_0 and η, i.e.,

$$\mathcal{T}[e^{i\theta}\phi_0, e^{-i\theta}\phi_0^*] = e^{i\theta}\mathcal{T}[\phi_0,\phi_0^*]. \tag{5.5}$$

Taking the gauge parameter θ infinitesimal and comparing the linear terms in θ in the above expression, we find to *all orders in perturbation theory* the Ward identity for the tadpole

$$\mathcal{T}[\phi_0,\phi_0^*] = \frac{\partial \mathcal{T}[\phi_0,\phi_0^*]}{\partial \phi_0}\ \phi_0 - \frac{\partial \mathcal{T}[\phi_0,\phi_0^*]}{\partial \phi_0^*}\phi_0^*. \tag{5.6}$$

Therefore combining (5.4) and the Ward identity (5.6), we obtain the following Ward identity which is an *exact* relationship between the tadpole and the self-energies at zero frequency and momentum

$$\phi_0\left[\mu - \Sigma_{11}(k=0,\omega=0)\right] + \phi_0^*\Sigma_{12}(k=0,\omega=0) = \mathcal{T}[\phi_0,\phi_0^*]. \tag{5.7}$$

Above the critical temperature *both* the tadpole and the condensate vanish, thus the above equation becomes a trivial identity. However, below the critical temperature $\phi_0 \neq 0$ and in equilibrium $\mathcal{T}[\phi_0,\phi_0^*]=0$, the above Ward identity (5.7) leads to the Hugenholtz-Pines relation

$$\mu = \Sigma_{11}(k=0,\omega=0) - \frac{\phi_0^*}{\phi_0}\Sigma_{12}(k=0,\omega=0). \tag{5.8}$$

It is customary to choose the condensate to be real by redefining its phase via the gauge transformation (2.14), in which case (5.8) leads to the familiar form of the Hugenholtz-Pines theorem. However, for a condensate with an arbitrary phase, the anomalous self-energy Σ_{12} *must be* proportional to ϕ_0^2 since in the equation of motion it multiplies δ^*, which transforms under gauge transformations just as ϕ_0^*. This fact can be seen explicitly at both the lowest and one-loop order in the expressions for the respective anomalous self-energy in (4.6) and in (4.22), (4.23) with the spectral functions given by (4.24). However, the product $\Sigma_{12}\delta^*$ must transform just as δ or ϕ_0 therefore the phase ϕ_0^*/ϕ_0 cancels the phase of ϕ_0^2 in Σ_{12}. In terms of the *gauge invariant* self-energies Σ_D and Σ_O the Hugenholtz-Pines theorem (5.8) becomes the more familiar form

$$\mu = \Sigma_D(k=0,\omega=0) - \Sigma_O(k=0,\omega=0). \tag{5.9}$$

We emphasize that while the Ward identity (5.7) is an exact relationship valid in or out of equilibrium (corresponding to when the tadpole $\mathcal{T}[\phi_0,\phi_0^*]$ vanishes or not, respectively), the Hugenholtz-Pines theorem (5.8) or alternatively (5.9) is *only* valid in equilibrium when the tadpole vanishes. This observation will become important when we discuss nonequilibrium issues below.

5.2 Confirmation at one-loop order

We now show explicitly that the Ward identity (5.6) is fulfilled to one-loop order. The tadpole up to one-loop order is given by (4.25). The Hartree-Fock-Bogoliubov (local) contributions to the one-loop self-energies are momentum and frequency independent and given by (4.20), leading to the relation

$$\Sigma_{11}^{\mathrm{HFB}} - \frac{\phi_0^*}{\phi_0}\Sigma_{12}^{\mathrm{HFB}} = g\left(n_0 + 2\widetilde{n} - \frac{\phi_0^*}{\phi_0}\widetilde{m}\right), \tag{5.10}$$

which obviously *do not* fulfill the Ward identity (5.6) with the tadpole up to one loop given by (4.25). This is a known fact that the Hartree-Fock-Bogoliubov approximation for the one-loop self-energies violates the Hugenholtz-Pines theorem.

However, using the explicit form of the nonlocal contributions for the self-energies given by (4.36) with the spectral functions given by (4.24), it is a tedious but straightforward exercise to find that (after analytic continuation to real frequency ω through the substitution $s \to -i\omega + 0^+$ [38,39])

$$\Sigma_{11}^{R}(k=0,\omega=0) - \frac{\phi_0^*}{\phi_0}\Sigma_{12}^{R}(k=0,\omega=0) = 2g\frac{\phi_0^*}{\phi_0}\widetilde{m}, \tag{5.11}$$

which is an *exact* relationship up to one-loop order, obtained for arbitrary μ and ϕ_0. This is a purely algebraic relation which we obtained without the use of either the dispersion relation of quasiparticles or of the lowest order relationship between ϕ_0 and μ implied by (4.8). Furthermore, we have checked explicitly that the above relation is independent of the order of limits (i.e., $k \to 0$ is taken either before or after $\omega \to 0$). While the individual contributions feature strong infrared divergences in the limit $k,\omega \to 0$ at finite temperature (see Ref. [8] for a discussion), they cancel exactly in the combination (5.11).

This cancellation is a manifestation of the general results of Gavoret and Nozières [24] at least for the combination that enters in the Ward identity.

When combined with (5.10) the above relation (5.11) leads to

$$\Sigma_{11}(k=0,\omega=0) - \frac{\phi_0^*}{\phi_0}\Sigma_{12}(k=0,\omega=0) = g\left(n_0 + 2\widetilde{n} + \frac{\phi_0^*}{\phi_0}\widetilde{m}\right). \tag{5.12}$$

Collecting the above results, we find up to one-loop order that

$$\begin{aligned}\frac{\mathcal{T}[\phi_0,\phi_0^*]}{\phi_0} &= \mu - \left[\Sigma_{11}(k=0,\omega=0) - \frac{\phi_0^*}{\phi_0}\Sigma_{12}(k=0,\omega=0)\right] \\ &= \mu - g\left(n_0 + 2\widetilde{n} + \frac{\phi_0^*}{\phi_0}\widetilde{m}\right).\end{aligned} \tag{5.13}$$

Therefore, the Ward identity (5.6) is manifestly fulfilled up to one-loop order by including the nonlocal parts of the self-energies. The equilibrium condition $\mathcal{T}[\phi_0,\phi_0^*] = 0$ with the relation (5.12) for the one-loop self-energies shows that the Hugenholtz-Pines condition (5.8) is fulfilled by properly including the *nonlocal* parts of the self-energies. This in turn implies a gapless spectrum for quasiparticle excitations [22,14,6].

This is an important advantage of the tadpole method of nonequilibrium field theory: the equations of motion obtained at a given order in the loop expansion are causal and guaranteed to fulfill the corresponding Ward identities to the given order. In the classification of Hohenberg and Martin [21] the tadpole method leads, therefore, to a *gapless* approximation that fulfills the Hugenholtz-Pines theorem.

6 Issues in and out of Equilibrium

6.1 Issues in equilibrium: nature of the perturbative expansion

A relevant question that emerges at this stage is the nature of the perturbative expansion. Taking the perturbative expansion in terms of the coupling g (or alternatively in terms of the T-matrix [36,8]) there seems to be an inconsistency in the calculation: the HFB local contributions to the self-energies as well as all of the tadpole contributions are formally of order $\mathcal{O}(g)$, while the nonlocal contributions are formally of order $\mathcal{O}(g^2)$. Obviously there seems to be a mismatch in the orders of coupling constant, but from the confirmation of the Ward identity at one-loop order, this mismatch is *required* to fulfill the Ward identity.

The failure of the perturbative expansion in terms of g can also be seen as follows. Using the lowest order Hugenholtz-Pines relation $\mu - gn_0 = 0$ [see (4.8)] so that for fixed chemical potential, according to the grand-canonical formulation, $gn_0 = \mu \sim \mathcal{O}(1)$. The nonlocal contributions to the self-energies [see (4.24)] are all of order $g(gn_0)$, therefore of order $\mathcal{O}(g)$, hence of the same order as the HFB contributions and the tadpole. This mismatch of powers of coupling constant and the subtleties associated with this had been recognized in Refs. [8,44].

As emphasized above with the explicit calculation, the framework that leads to a consistent expansion, at least for temperatures well below the critical, is the *loop expansion*. The loop expansion is formally an expansion in powers of $\hbar$ and *not* of g. The basis for the loop expansion is that powers of $\hbar$ describe quantum corrections, which at finite temperature are mixed with thermal contributions and can be systematically implemented as follows.

Firstly the Lagrangian should be formally considered to be independent of $\hbar$, i.e., *classical*. Although the coupling g depends on $\hbar$ through its relation with the s-wave scattering length, a systematic implementation of the loop expansion requires that $\mathcal{L} \sim \mathcal{O}(1)$ in terms of $\hbar$ and hence g must be understood as a parameter in the Lagrangian of order $\mathcal{O}(1)$. In the path integral representation, each path is weighted by the exponential of the action divided by $\hbar$, so in the path integral the Lagrangian enters in the form $\sim \exp\left(\frac{i}{\hbar}\int d^4x\mathcal{L}\right)$. The $\hbar$ can be absorbed in a redefinition of the quantum fields

$$\chi, \chi^\dagger \to \sqrt{\hbar}\chi, \sqrt{\hbar}\chi^\dagger. \tag{6.1}$$

The path integral measure is multiplied by an overall factor which cancels out in all correlation functions.

Writing the Lagrangian as in (2.12) and gathering the terms linear, quadratic, cubic and quartic in χ, $\chi^\dagger$ in $\mathcal{L}_{\rm int}$ as

$$\mathcal{L}_{\rm int} = \mathcal{L}^{(1)}_{\rm int} + \mathcal{L}^{(2)}_{\rm int} + \mathcal{L}^{(3)}_{\rm int} + \mathcal{L}^{(4)}_{\rm int}, \tag{6.2}$$

then upon rescaling the fields as in (6.1) one finds

$$\frac{\mathcal{L}}{\hbar} \to \mathcal{L}_0 + \frac{1}{\sqrt{\hbar}}\mathcal{L}^{(1)}_{\rm int} + \mathcal{L}^{(2)}_{\rm int} + \sqrt{\hbar}\mathcal{L}^{(3)}_{\rm int} + \hbar\mathcal{L}^{(4)}_{\rm int}. \tag{6.3}$$

It is now straightforward to see that (i) the lowest order term in the equation of motion (4.4) are of order $\mathcal{O}(1/\sqrt{\hbar})$ since they arise from the linear term $\mathcal{L}^{(1)}_{\rm int}$ in the Lagrangian, (ii) the tadpole and the HFB contributions to the one-loop self-energies are obtained from $\sqrt{\hbar}\mathcal{L}^{(3)}_{\rm int}$ and are therefore of order $\mathcal{O}(\sqrt{\hbar})$, and (iii) the nonlocal contributions are obtained by one insertion of $\mathcal{L}^{(3)}_{\rm int}$ and one insertion of $\mathcal{L}^{(2)}_{\rm int}$ and are therefore *also* of order $\mathcal{O}(\sqrt{\hbar})$. Hence, the tadpole, HFB and nonlocal contributions to the self-energies are *all* of order $\mathcal{O}(\sqrt{\hbar})$, therefore $\mathcal{O}(\hbar)$ as compared to the lowest order (classical) contributions. As a result of this counting, diagrams with n-loops are multiplied by $\hbar^n$ with respect to the classical terms in the equations of motion. Thus when the tadpole and the self-energies are computed to the same order in the $\hbar$ expansion the Ward identity is guaranteed to be fulfilled. While the identification of the loop expansion with a power series expansion in $\hbar$ is well known in the non-condensed phase, the presence of new interaction vertices proportional to the condensate ϕ_0 in the condensed phase introduces different powers of $\hbar$ in the rescaled Lagrangian. The necessity for replacing the expansion in terms of the coupling g by a loop expansion in terms of quantum (and thermal) corrections was first recognized in Ref. [44], but to our knowledge the consistency of the loop expansion for fulfilling the Ward identity has not been highlighted in detail before.

The consistency of the loop expansion still requires the resolution of another important issue, which has already been recognized in Ref. [44]. In equilibrium and for a fixed chemical potential, the equilibrium value of the condensate is determined by the equilibrium condition $\mathcal{T}[\phi_0, \phi_0^*] = 0$. As argued above, the expression for the tadpole $\mathcal{T}[\phi_0, \phi_0^*]$ can be recognized as the δ independent terms in the equations of motion in the loop expansion. Since the tadpole is obtained as a power expansion in $\hbar$, the equilibrium value of the condensate ϕ_0 can also be found as an expansion in $\hbar$. However, in the diagonalization of the quadratic part of the Lagrangian $\mathcal{L}_0$, the condensate ϕ_0 enters as a parameter. As a result, the Bogoliubov coefficients u_k, v_k and the dispersion relation of quasiparticles ω_k depend implicitly on $\hbar$ through ϕ_0 (see the Appendix). Thus keeping the full ϕ_0 in the free quasiparticle Lagrangian $\mathcal{L}_0$ and eventually identifying ϕ_0 with the solution of the equilibrium condition $\mathcal{T}[\phi_0, \phi_0^*] = 0$ in the loop expansion, leads to an inconsistent expansion in $\hbar$.

The remedy for this problem is to expand the full ϕ_0 in powers of $\hbar$

$$\phi_0 = \overline{\phi}_0 + \Delta, \tag{6.4}$$

where $\overline{\phi}_0 \sim \mathcal{O}(\hbar^0)$ and $\Delta = \hbar\Delta^{(1)} + \hbar^2\Delta^{(2)} + \cdots$. Thus in the free quasiparticle Lagrangian $\mathcal{L}_0$ we now replace ϕ_0 by the zeroth order term $\overline{\phi}_0$, while the contributions that include Δ are now lumped together in the interaction Lagrangian $\mathcal{L}_{\text{int}}$. The new interaction Lagrangian now becomes

$$
\begin{aligned}
\mathcal{L}_{\rm int}[\chi,\chi^\dagger] = \chi^\dagger \Bigg[& \left(i\frac{\partial}{\partial t} + \frac{\nabla^2}{2m} + \mu - 2g|\phi_0|^2 \right) \delta - g\phi_0^2 \delta^* + \phi_0 \left(\mu - g|\phi_0|^2 \right) \\
& + \eta \Bigg] - g\left(\overline{\phi}_0 \Delta^* + \overline{\phi}_0^* \Delta + |\Delta|^2 \right) \chi^\dagger \chi - \frac{g}{2} \left(2\overline{\phi}_0 + \Delta^2 \right) \chi^\dagger \chi^\dagger \\
& - 2g\phi_0^* \delta \chi^\dagger \chi - g\phi_0 \delta \chi^\dagger \chi^\dagger - g\phi_0 \chi \chi^\dagger \chi^\dagger - g\delta \chi \chi^\dagger \chi^\dagger - \frac{g}{2} \chi^\dagger \chi^\dagger \chi \chi \\
& + \mathrm{H.c.},
\end{aligned}
\tag{6.5}
$$

where $\phi_0 = \overline{\phi}_0 + \Delta$ in all the terms that involve ϕ_0.

In this manner, a perturbative expansion around the equilibrium condensate now in terms of the free quasiparticle Lagrangian that only involves $\overline{\phi}_0$ and interactions that are systematically treated in the loop ($\hbar$) expansion is consistently cast in terms of the gapless free Bogoliubov quasiparticles as the noninteracting states. The equation of motion, now consistently obtained in the loop expansion will involve a tadpole and self-energies that are expanded to the same order in $\hbar$, thus guaranteeing that the Ward identity is fulfilled and in equilibrium the quasiparticle excitations are gapless by the Hugenholtz-Pines theorem. A straightforward calculation reveals that to $\mathcal{O}(\hbar)$ our calculation is consistent by using $\overline{\phi}_0$ in the Bogoliubov coefficients and dispersion relations in the propagators. The resulting tadpole equation can be systematically solved up to $\mathcal{O}(\hbar)$ and the self-energies are unambiguously of order $\mathcal{O}(\hbar)$ with respect to the classical term. The Ward identity is fulfilled exactly in the manner displayed in the previous section.

Near the critical temperature the loop expansion fails because of the presence of infrared divergences. Either renormalization group [44,45] or other nonperturbative techniques [46,47] must be invoked for a systematic study of both the static and the dynamical aspects near the critical point, which of course fall outside the scope of this article.

6.2 Issues out of equilibrium: condensate instabilities

When the condensate is in equilibrium, i.e., $\mathcal{T}[\phi_0,\phi_0^*]=0$, the solution of the equation of motion for small amplitude perturbations around equilibrium will reveal the frequencies and damping rates of the quasiparticle excitations of the condensate. However, the derivation of the equations of motion allows a more general study and in particular to address the question of dynamics *away* from equilibrium. If the tadpole $\mathcal{T}[\phi_0,\phi_0^*]\neq 0$, the homogeneous condensate ϕ_0 does *not* describe a situation of equilibrium. This can be seen, for example, from the equations of motion (4.17), which even for $\eta_{\mathbf{k}}(t)=0$ features a nonvanishing inhomogeneity $\mathcal{T}[\phi_0,\phi_0^*]$.

To understand the nonequilibrium aspects, let us first focus on the spectrum of quasiparticle excitations at *lowest* order. The excitation energies of free Bogoliubov quasiparticles ω_k given by (3.31) become *imaginary* for long wavelengths $k \approx 0$ when

$$(\mu - g|\phi_0|^2)(\mu - 3g|\phi_0|^2) < 0. \tag{6.6}$$

These imaginary frequencies correspond to instabilities of the homogeneous Bose gas for the values of chemical potential μ and condensate ϕ_0 for which the condition (6.6) is fulfilled. These modes cannot be treated as stable quasiparticles and their treatment has to be modified, in particular no occupation number can be defined for these modes. The equations of motion in this situation describes a nonvanishing force term proportional to $\mathcal{T}[\phi_0, \phi_0^*]$ that makes the long wavelength condensate perturbations to evolve in time. The instabilities for long wavelength excitations reveal the growth of the condensate perturbations [29,48] and are qualitatively similar to spinodal decomposition in self-interacting field theories [49], which must be studied self-consistently and nonperturbatively. This is a topic that we are currently study with the formulation presented in this article and on which we expect to report soon.

7 Conclusions

In this article we have focused on establishing a program to obtain the equations of motion for small amplitude perturbations of a homogeneous condensate based on linear response in a manner consistent with the underlying symmetries. We introduced a method that combines the Schwinger-Keldysh formulation of nonequilibrium quantum field theory, the Nambu-Gor'kov formalism of the Bogoliubov quasiparticle excitations in the condensed phase and a novel technique, the tadpole method, used in relativistic field theory to obtain the equations of motion directly in real time and consistently in perturbation theory. This method automatically leads to causal equations of motion and allow to extract the Ward identities associated with the underlying gauge symmetry that are valid in or out of equilibrium. In equilibrium, as can be determined by a definite condition, these Ward identities lead to the Hugenholtz-Pines theorem, therefore this method leads to equations of motion that guarantee gapless quasiparticle excitations of the condensate.

Furthermore, this method leads to a formulation of the time evolution of perturbations away from the equilibrium condensate as an initial value problem, thereby establishing contact with potential experimental situations in which a particular initial state is prepared away from equilibrium. It also highlights that the consistent perturbative expansion is *not* an expansion in the coupling, but a loop expansion, at least at low temperature. The loop expansion can be systematically implemented and is guaranteed to fulfill the Ward identities order by order in perturbation theory. We have obtained the equations of motion consistently to one-loop order and showed explicitly that the inclusion of nonlocal (absorptive) contributions to the self-energies that are beyond the Hartree-Fock-Bogoliubov approximation and correspond to the Beliaev and Landau damping processes are necessary to the fulfillment of the Ward identities in or out of equilibrium.

While our focus in this article is to present the formulation that leads to causal equations of motion which fulfill the Ward identities and to explore some of their consequences, our more overarching goal is to study the real-time dynamics of nonequilibrium states directly in real time both in homogeneous and inhomogeneous (trapped) condensates. The homogeneous case studied here may not bear much relevance to the experimental situation, however, the techniques presented in this article can be extrapolated almost immediately to the inhomogeneous case of trapped atomic gases. Even in the presence of a trapping one-body potential, the underlying gauge symmetry is global, i.e., the gauge transformations that leave the Lagrangian invariant are space independent. However, an important modification will be that the condensate becomes localized in the center of the trapping potential and hence breaks translational invariance. This also implies that the tadpole $\mathcal{T}[\phi_0(\mathbf{x}), \phi_0^*(\mathbf{x})]$ now acquires implicit space dependence through the inhomogeneous condensate $\phi_0(\mathbf{x})$, which in turn will require a modification of the steps that lead to the Ward identities. While all of these issues require a careful study, we expect that the method presented here will allow us to extract the relevant Ward identities in the inhomogeneous case as well.

We are currently studying these aspects with particular attention to the possible infrared divergences and their potential impact on dynamics away from equilibrium and instabilities of the condensates. We expect to report on many of these issues in a forthcoming article.

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A Plane wave solutions and the Bogoliubov transformation

In this Appendix we present an alternative derivation of the correlation functions for χ, $\chi^\dagger$ directly from the plane wave solutions of the homogeneous equations of motion (i.e., in the absence of source) for the Nambu-Gor'kov field given by (3.15)

$$\left[i\sigma_3\frac{\partial}{\partial t}+\frac{\nabla^2}{2m}+\mu-2g|\phi_0|^2-g\phi_0^2\sigma_+-g\phi_0^{*2}\sigma_-\right]\Psi(\mathbf{x},t)=0. \tag{A.1}$$

The plane wave solution can be written in the form

$$\Psi(\mathbf{x},t)=S(k)\,e^{-i(E_kt-\mathbf{k}\cdot\mathbf{x})},\quad S(k)=\begin{bmatrix}U_k\\ V_k\end{bmatrix}, \tag{A.2}$$

where the two-component spinor obeys

$$\begin{bmatrix}\xi_k & g\phi_0^2\\ g\phi_0^{*2} & \xi_k\end{bmatrix}\begin{bmatrix}U_k\\ V_k\end{bmatrix}=E_k\,\sigma_3\begin{bmatrix}U_k\\ V_k\end{bmatrix}, \tag{A.3}$$

where $\xi_k=k^2/2m-\mu+2g|\phi_0|^2$. This is an eigenvalue equation with a "weight" σ_3. The eigenvalues are given by $E_k=\pm\omega_k$ with $\omega_k=\sqrt{\xi_k^2-(g|\phi_0|^2)^2}$. The normalization of the positive and negative energy spinors is chosen so that

$$S^{(\alpha)\dagger}(k)\sigma_3S^{(\beta)}(k)=(\sigma_3)_{\alpha\beta}, \tag{A.4}$$

where $\alpha,\beta=1,2$ (not to be confused with the components of the spinors) correspond to $E_k=\pm\omega_k$, respectively. That this is the correct choice stems from σ_3 being the "weight" in the eigenvalue equation as well as comparing with the limit $\phi_0=0$ which is the free field case. Introducing the Bogoliubov coefficients u_k and v_k

$$\begin{aligned}u_k&=\left(\frac{\omega_k+\xi_k}{2\omega_k}\right)^{1/2},\quad v_k=-u_k\left(\frac{g\phi_0^{*2}}{\omega_k+\xi_k}\right),\\ u_kv_k&=-\frac{g\phi_0^{*2}}{2\omega_k},\quad \frac{v_k}{u_k}=-r_k\frac{\phi_0^*}{\phi_0},\quad r_k=\frac{g|\phi_0|^2}{\omega_k+\xi_k},\end{aligned} \tag{A.5}$$

we find that the positive and negative energy spinors are given by

$$S^{(1)}(k) = \begin{bmatrix} u_k \\ v_k \end{bmatrix}, \quad S^{(2)}(k) = \begin{bmatrix} v_k^* \\ u_k \end{bmatrix}, \tag{A.6}$$

respectively. After accounting for the interpretation of negative energy solutions as antiparticles, one can therefore write the general plane wave solution of the homogeneous equation of motion (A.1) as

$$\Psi(\mathbf{x}, t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \left\{ \alpha_{\mathbf{k}} \begin{bmatrix} u_k \\ v_k \end{bmatrix} e^{-i(\omega_k t - \mathbf{k}\cdot\mathbf{x})} + \alpha_{\mathbf{k}}^{\dagger} \begin{bmatrix} v_k^* \\ u_k \end{bmatrix} e^{i(\omega_k t - \mathbf{k}\cdot\mathbf{x})} \right\}, \tag{A.7}$$

where V is the quantization volume.

At the level of second quantization, one recognizes that (A.7) is the Bogoliubov transformation. The operators $\alpha_{\mathbf{k}}$ and $\alpha_{\mathbf{k}}^{\dagger}$, respectively, annihilate and create a Bogoliubov quasiparticle of momentum $\mathbf{k}$ (energy ω_k) and obey the usual canonical commutation relations. The correlation functions of the Nambu-Gor'kov fields in the density matrix that describes free Bogoliubov quasiparticles in thermal equilibrium at inverse temperature β are therefore found to be given by

$$\begin{aligned} \langle \Psi_a(\mathbf{x}, t) \Psi_b^{\dagger}(\mathbf{x}', t') \rangle = & \frac{1}{V} \sum_{\mathbf{k}} \Big[[1 + n_B(\omega_k)] \mathcal{G}_{ab}(k) e^{-i\omega_k(t-t')} \\ & + n_B(\omega_k) \overline{\mathcal{G}}_{ab}(k) e^{i\omega_k(t-t')} \Big] e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}, \\ \langle \Psi_b^{\dagger}(\mathbf{x}', t') \Psi_a(\mathbf{x}, t) \rangle = & \frac{1}{V} \sum_{\mathbf{k}} \Big[n_B(\omega_k) \mathcal{G}_{ab}(k) e^{-i\omega_k(t-t')} \\ & + [1 + n_B(\omega_k)] \overline{\mathcal{G}}_{ab}(k) e^{i\omega_k(t-t')} \Big] e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}, \end{aligned} \tag{A.8}$$

where $n_B(\omega_k)$ is the equilibrium distribution for Bogoliubov quasiparticles of momentum $\mathbf{k}$

$$n_B(\omega_k) = \langle \alpha_{\mathbf{k}}^{\dagger} \alpha_{\mathbf{k}} \rangle = \frac{1}{e^{\beta\omega_k} - 1}, \tag{A.9}$$

and $\mathcal{G}(k)$, $\overline{\mathcal{G}}(k)$ are given by (3.30).

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