Date: Saturday 07 May 2005  
Instructor: Dr. A. Mekki  
Time of the exam: 90 minutes

Name: ___________________________ Key: ___________________________ Id#: ___________________________

SHOW THE DETAILS OF YOUR WORK

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Q1. (5 points)
Consider a proton moving with a speed of $3 \times 10^7$ m/s along the x-axis. If the uncertainty in its position is $0.01 \times 10^{-10}$ m, calculate the minimum uncertainty in
(a) its momentum

\[ v = 3.0 \times 10^7 \text{ m/s} \quad \Delta x = 0.01 \times 10^{-10} \text{ m} \]

\[ \Delta \mathbf{p}_x = \frac{\hbar}{\Delta x} = \frac{1.05 \times 10^{-34}}{2 \times 0.01 \times 10^{-10}} = 5.25 \times 10^{-23} \text{ kg} \cdot \text{m/s} \]

(b) its speed

\[ p_x = m v_x \Rightarrow \Delta p_x = m \Delta v_x \Rightarrow \Delta v_x = \frac{\Delta p_x}{m} \]

\[ \Rightarrow \Delta v_x = \frac{5.25 \times 10^{-23}}{1.67 \times 10^{-27}} = 3.14 \times 10^4 \text{ m/s} \]

(c) its kinetic energy

\[ K = \frac{1}{2} m v^2 \Rightarrow \Delta K = \frac{1}{2} 2 m v \Delta v \]

\[ \Rightarrow \Delta K = m v \Delta v = 1.67 \times 10^{-27} \times 3 \times 10^7 \times 3.14 \times 10^4 \]

\[ \Rightarrow \Delta K = 1.58 \times 10^{-15} \text{ J} \]
Q2. (5 points)
Calculate the de Broglie wavelength of an electron accelerated through a potential difference of 5 MV.
Hint: treat the problem relativistically, that is, use the relativistic total and kinetic energies of the electron.

\[
\lambda = \frac{\hbar}{p}
\]

\[
E^2 = p^2c^2 + m_0^2c^4 = \left( K + m_0c^2 \right)^2
\]

\[
= K^2 + 2Km_0c^2 + m_0^2c^4
\]

\[
\Rightarrow p c = \sqrt{K^2 + 2Km_0c^2} = K \sqrt{1 + \frac{2m_0c^2}{K}}
\]

\[
p = \frac{Kc}{\sqrt{1 + \frac{2m_0c^2}{K}}}
\]

\[
\Rightarrow \lambda = \frac{\hbar}{p} = \frac{\hbar c}{K} \left( 1 + \frac{2m_0c^2}{K} \right)^{-1/2} \text{ and } K = eV
\]

\[
\lambda = \frac{\hbar c}{eV} \left( 1 + \frac{2m_0c^2}{eV} \right)^{-1/2}
\]

\[
= \frac{12400 \text{ eVÅ}}{5 \times 10^6 \text{ eV}} \left( 1 + \frac{2 \times 511 \text{ eV}}{5 \times 10^6 \text{ eV}} \right)^{-1/2}
\]

\[
\lambda = 2.3 \times 10^{-3} \text{ Å}
\]
Q3. (10 points)
An electron is trapped in a one-dimensional region of length \(1.0 \times 10^{-10}\) m (a typical atomic diameter).
(a) How much energy (in eV) must be supplied to excite the electron from the ground state to the third excited state?

\[
\Delta E = E_4 - E_1 = \frac{\pi^2 \hbar^2}{2mL^2} \left( \frac{4}{L} \right)^2 - \left( \frac{1}{L} \right)^2 = \frac{15 \pi^2 \hbar^2}{2mL^2} = 8.9 \times 10^{-17} \text{ J} = 561 \text{ eV}
\]

(b) In the third excited state, what is the probability of finding the electron in the region from \(x = 0.1 \times 10^{-10}\) m to \(x = 0.2 \times 10^{-10}\) m?

\[
P = \int_{0.1\text{nm}}^{0.2\text{nm}} |\psi|^{2} \, dx = \frac{2}{L} \int_{0.1L}^{0.2L} \sin^{2}\left(\frac{4\pi}{L} x\right) \, dx
\]

\[
= \frac{1}{L} \left[ \frac{1 - \cos\left(\frac{8\pi}{L} x\right)}{2} \right]_{0.1L}^{0.2L}
= \frac{1}{L} \left[ \frac{1}{2} \left[ \sin\left(\frac{8\pi}{L} \cdot 0.2L\right) - \sin\left(\frac{8\pi}{L} \cdot 0.1L\right) \right] \right]
= \frac{1}{L} \left[ \frac{1}{2} \left[ \sin(1.6\pi) - \sin(0.8\pi) \right] \right]
= 0.16 = 16\%
\]

(c) Draw a diagram showing the probability density for the third excited state and deduce the probability of finding the electron in the region \(x = 0\) and \(x = 0.75\times10^{-10}\) m.

\[
P = \frac{3}{4} = 0.75 = 75\%
\]
Q4. (5 points)
(a) Determine whether each of the following functions is acceptable or not as a wavefunction over the indicated interval and explain the reason for those not acceptable.

(a) $\Psi(x)$, Not smooth
(b) $\Psi(x)$, Not single valued
(c) Yes.
(d) Yes.
(e) No, Not continuous
Q5. (10 points)
Consider the wavefunction of the ground state of a simple harmonic oscillator given by
\[ \psi_0(x) = \left( \frac{\alpha}{\pi} \right)^{1/4} e^{-\alpha x^2/2} \] where \( \alpha = \frac{m \omega}{\hbar} \). Calculate
(a) \( \langle x \rangle \)
\[
\langle x \rangle = \int_{-\infty}^{+\infty} x |\psi_0|^2 \, dx = \left( \frac{\alpha}{\pi} \right)^{1/2} \int_{-\infty}^{+\infty} x e^{-\alpha x^2} \, dx = 0
\]
(b) \( \langle x^2 \rangle \)
\[
\langle x^2 \rangle = \int_{-\infty}^{+\infty} x^2 |\psi_0|^2 \, dx = \left( \frac{\alpha}{\pi} \right)^{1/2} \int_{-\infty}^{+\infty} x^2 e^{-\alpha x^2} \, dx
\]
\[
= \left( \frac{\alpha}{\pi} \right)^{1/2} \cdot \frac{1}{4\alpha} \left( \frac{\pi}{\alpha} \right)^{1/2}
\]
\[
= \frac{1}{4\alpha} = \frac{\hbar}{4m\omega}
\]
(c) \( \langle p \rangle \)
\[
\langle p \rangle = -i\hbar \left( \frac{\pi}{\alpha} \right)^{1/2} \int_{-\infty}^{+\infty} e^{-\alpha x^2/2} \frac{\partial}{\partial x} \left( e^{-\alpha x^2/2} \right) \, dx
\]
\[
= -i\hbar \left( \frac{\pi}{\alpha} \right)^{1/2} \int_{-\infty}^{+\infty} e^{-\alpha x^2/2} (-\alpha x) e^{-\alpha x^2/2} \, dx
\]
\[
= i\alpha\hbar \left( \frac{\pi}{\alpha} \right)^{1/2} \int_{-\infty}^{+\infty} x e^{-\alpha x^2} \, dx = 0
\]
(d) \( \langle p^2 \rangle \)

\[
\langle k \rangle + \langle u \rangle = \langle E \rangle
\]

\[
\frac{\langle p^2 \rangle}{2m} + \frac{1}{2}k \langle x^2 \rangle = \frac{1}{2} \hbar \omega; \quad \text{but } k = \frac{m \omega^2}{2}
\]

\[
\frac{\langle p^2 \rangle}{2m} + \frac{1}{2} \omega \hbar \frac{h}{2m \hbar} = \frac{1}{2} \hbar \omega
\]

\[
\frac{\langle p^2 \rangle}{2m} = \frac{1}{2} \hbar \omega - \frac{1}{4} \hbar \omega = \frac{1}{4} \hbar \omega
\]

\[
\Rightarrow \quad \langle p^2 \rangle = \frac{1}{2} m \hbar \omega
\]

Another method is \( \langle p^2 \rangle = h^2 \left( \frac{\alpha}{\pi} \right)^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\frac{\alpha}{2} x^2} \frac{d^2}{dx^2} e^{-\frac{\alpha}{2} x^2} dx \)

(e) \( \Delta x \cdot \Delta p \)

\[
\Delta x = \sqrt{\langle x^2 \rangle - \langle x^2 \rangle^2} = \sqrt{\frac{\hbar}{2m \omega}}
\]

\[
\Delta p = \sqrt{\langle p^2 \rangle - \langle p^2 \rangle^2} = \sqrt{\frac{1}{2} m \hbar \omega}
\]

\[
\Delta x \cdot \Delta p = \sqrt{\frac{\hbar^2}{4}} = \frac{\hbar}{2}
\]

(f) Is the uncertainty principle violated? Explain.

\[
\text{No, since } \Delta x \cdot \Delta p \geq \frac{\hbar}{2} \text{ is satisfied}
\]
Q6. (5 points)
Consider an electron held in a two dimensional infinite potential well in the form

\[
U(x,y) = \begin{cases} 
0 & 0 < L, 0 < y < 2L \\
\infty & \text{otherwise}
\end{cases}
\]

Find the energies of the four lowest states and their corresponding normalized wavefunctions. State which of these states are degenerate.

\[
E_{n_1,n_2} = \frac{\pi^2 \hbar^2}{2m} \left( \frac{n_1^2}{L^2} + \frac{n_2^2}{4L^2} \right) = \frac{E_0}{8mL^2} \left( 4n_1^2 + n_2^2 \right)
\]

- Ground state: \( n_1 = n_2 = 1 \) \( E_{11} = 5E_0 \)
- First excited state: \( n_1 = 1 \) \( n_2 = 2 \) \( E_{12} = 8E_0 \)
- Second excited state: \( n_1 = 1 \) \( n_2 = 3 \) \( E_{13} = 13E_0 \)
- Fourth excited state: \( n_1 = 2 \) \( n_2 = 1 \) \( E_{21} = 17E_0 \)

The wavefunctions are:

\[
\psi_{11} = \frac{\sqrt{2}}{L} \sqrt{\frac{2}{2L}} \sin \left( \frac{\pi x}{L} \right) \sin \left( \frac{\pi y}{2L} \right)
\]

\[
= \frac{\sqrt{2}}{L} \sin \left( \frac{\pi x}{L} \right) \sin \left( \frac{\pi y}{2L} \right)
\]

\[
\psi_{12} = \frac{\sqrt{2}}{L} \sin \left( \frac{\pi x}{L} \right) \sin \left( \frac{\pi y}{2L} \right)
\]

\[
\psi_{13} = \frac{\sqrt{2}}{L} \sin \left( \frac{\pi x}{L} \right) \sin \left( \frac{3\pi y}{2L} \right)
\]

\[
\psi_{21} = \frac{\sqrt{2}}{L} \sin \left( \frac{2\pi x}{L} \right) \sin \left( \frac{\pi y}{2L} \right)
\]

None of these states are degenerate.
Q7. (10 points)
The radial part of the wavefunction for the hydrogen atom in the $2p$ state is given by

$$R_{2p} = \left(\frac{1}{2a_0}\right)^{3/2} \frac{r}{a_0 \sqrt{3}} e^{-r/2a_0}$$

where $a_0$ is Bohr radius.

(a) Find the most probable distance of the electron from the proton when the electron is in this state.

$$P_{2p} = r^2 R_{2p} = \left(\frac{1}{2a_0}\right)^{3/2} \frac{r^4}{a_0^2} e^{-r/a_0}$$

most probable distance $\Rightarrow \frac{dP_{2p}}{dr} = 0 \quad (P_{2p} \text{ is maximum})$

$$\frac{d}{dr} \left( r^4 e^{-r/a_0} \right) = 4r^3 e^{-r/a_0} - \frac{r^4}{a_0} e^{-r/a_0} = 0$$

$$(4 - \frac{r}{a_0}) r^3 e^{-r/a_0} = 0 \Rightarrow r = 0 \quad \text{(impossible)} \quad \Rightarrow r = 4a_0$$

(b) Calculate the average value of $r$ when the electron is in this state.

$$\langle r \rangle = \int_0^{+\infty} r P_{2p}(r) \, dr = \left(\frac{1}{2a_0}\right)^{3/2} \frac{1}{3a_0^2} \int_0^{+\infty} r^5 e^{-r/a_0} \, dr$$

set $y = \frac{r}{a_0} \Rightarrow r = a_0 y \, dy = a_0 \, dz$

$$\Rightarrow \langle r \rangle = \frac{1}{\left(2a_0\right)^{3/2}} \frac{1}{3a_0^2} \int_0^{+\infty} a_0^5 \frac{y^5}{a_0^5} e^{-y} \, dz = \frac{a_0}{2} \left\{ \int_0^{+\infty} y^5 e^{-y} \, dy \right\}$$

$$\langle r \rangle = \frac{120}{24} a_0 = 5a_0$$

$$\langle r \rangle = 5a_0$$

(c) Explain briefly why is the most probable distance is different from the average value?

The function $P(r)$ is not symmetric as seen from the figure.