SOLUTION TO FIRST EXAMINATION.

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1. Fill in the following table with the words TRUE or FALSE where appropriate.

Differential Equation	separable	exact	linear 1st order ode
$(x^2 - y^2)dx + (e^y \ln y - 2xy)dy = 0$	FALSE	TRUE	FALSE
$xdy - (2y + x^3 \cos x)dx = 0$	FALSE	FALSE	TRUE
$\frac{dy}{dx} = a^{x+y} \quad (a > 0, a \neq 1)$	TRUE	TRUE	FALSE

2. Consider the initial value problem (IVP)

$$y' = \sqrt{y^2 - 9}$$
$$y(x_0) = y_0.$$

Describe (x_0, y_0) for which the IVP will have unique solution

SOLUTION. $f(x, y) = \sqrt{y^2 - 9}$ is continuous in the region

$$\{(x, y) : x \in \mathbb{R},\ y^2 \geq 9\} = \{(x, y) : x \in \mathbb{R},\ |y| \geq 3\}$$

But $\dfrac{\partial y}{\partial x} = \dfrac{y}{\sqrt{y^2 - 9}}$ is continuous in the region

$$\{(x, y) : x \in \mathbb{R},\ y^2 > 9\} = \{(x, y) : x \in \mathbb{R},\ |y| > 3\}$$

Therefore, the given IVP has unique solution through all points (x_0, y_0) such that $x_0 \in \mathbb{R}$ and $|y_0| > 3$. This is the region

$$\{(x, y) :\ x \in \mathbb{R},\ y > 3\} \cup \{(x, y) : x \in \mathbb{R},\ y < -3\}$$

3. Obtain a one parameter family of solutions to the differential equation

$$(1+y^2)dx + \sqrt{1-x^2}dy = 0$$

Your solution must be displayed in the *explicit* form.

SOLUTION. Separate the variables to get

$$\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{1+y^2} = 0$$

Integrate to get

$$\sin^{-1} x + \tan^{-1} y = c$$

where c is an arbitrary constant. This solution can be expressed explicitly as

$$y(x) = \tan(c - \sin^{-1} x).$$

4. If

$$y = f(x; c_1, c_2, c_3)$$

where, c_1, c_2, c_3 are arbitrary constants, is the solution of a linear differential equation, then the differential equation must be of order <u>THREE</u>. This solutions is of the <u>EXPLICIT</u> form.

5. Let a be a non-zero constant. Solve

$$y + xy' = a(1 + xy).$$

SOLUTION. Open up the bracket to get $y + xy' = a + axy$. Then collect together the terms in y to get $xy' + (1 - ax)y = a$. This is then put in the standard form as

$$y' + \left(\frac{1}{x} - a\right) y = \frac{a}{x}.$$

Therefore the integrating factor

$$\begin{aligned} \mu &= e^{\int\left(\frac{1}{x}-a\right)dx} \\ &= e^{\ln|x|-ax} = |x|e^{-ax} = |x|e^{-ax} \end{aligned}$$

Choose $\mu = xe^{-ax}$ and then multiply the standard form with μ to get

$$xe^{-ax}y' + xe^{-ax}\left(\frac{1}{x} - a\right)y = ae^{-ax}.$$

That is, $\left(xe^{-ax}y\right)' = ae^{-ax}$ which, on integration, gives $xe^{-ax}y = c - e^{-ax}$ where c is an arbitrary constant of integration. Therefore,

$$y = \frac{c}{x}e^{ax} - \frac{1}{x}, \qquad x \neq 0$$

6. Obtain the critical points of the differential equation

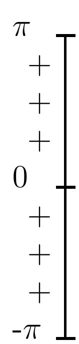
$$\frac{dy}{dx} = y \sin y$$

Given that $y = 0$ is a critical solution, determine whether it is stable, semi-stable or unstable

SOLUTION The critical points are given by $c \sin c = 0$. This gives $c = 0$ or $\sin c = 0$. The latter yields $c = n\pi,\ n = 0, \pm 1, \pm 2, \ldots$. Thus, the critical points are

$$c = 0,\ 0,\ \pm\pi,\ \pm 2\pi, \ldots$$

Now consider the equilibrium (critical) solution $y_c(x) = 0$. When $-\pi < y < 0$, then $y' = y \sin y > 0$ and y is an increasing function. Similarly, when $0 < y < \pi$, $y' = y \sin y > 0$ and and again y is an increasing function. Thus $y_c(x) = 0$ is an attractor for $y < 0$ and a repeller for $y > 0$. Therefore, $y_c(x) = 0$ is *semi-stable.*



Phase Portrait Around $c = 0$