Investigation of the constraints on harmonic morphisms of warped product type from Einstein manifolds

Submitted by

Dr. Hassan Azad

Department of Mathematical Sciences, KFUPM

and

Dr. Muhammad Tahir Mustafa

Department of Mathematical Sciences, KFUPM

for

Funding under Fast Track Research Program

TABLE OF CONTENTS

Abstract

1.0 Introduction

2.0 Literature review

3.0 Objectives of the study

4.0 Methodology

5.0 Management Plan

6.0 Significance of the study

7.0 Scheduling of work

8.0 References

9.0 Budget

10.0 Suggested Referees

ABSTRACT

Harmonic morphisms are maps between Riemannian manifolds which preserve germs of harmonic functions. These can be described as the harmonic maps which are horizontally (weakly) conformal.

A class of harmonic morphisms directly related to a physically significant geometric structure is the class of harmonic morphisms of warped product type. Such maps are characterized as non-constant horizontally homothetic harmonic morphisms with totally geodesic fibres and integrable horizontal distribution.

An investigation of harmonic morphisms of warped product type from Einstein manifolds is contained in a series of recent papers [Pantilie 2002], [Pantilie and Wood 2002a, 2002b]. In another recent publication, the book [Baird and Wood 2003; Proposition 12.7.1], it is shown that there are no non-trivial harmonic morphisms of warped product type (with one-dimensional fibres) from compact Einstein manifolds.

Motivated by the above result and the fact that there are natural obstructions to the existence of harmonic morphisms from compact domains, the aim of this project is to investigate constraints on the existence of harmonic morphisms of warped product type (with fibres of any dimensions) from compact Einstein manifolds. Precisely, the focus of the work will be to look into the following question.

> *Are there any harmonic morphisms of warped product type from compact Einstein manifolds? If yes, what are the restrictions on their existence?*

A successful completion of the project will also lead to (partial or complete) answer to the longstanding open question about the existence of compact Einstein warped products, posed by Besse in his famous book "Einstein manifolds" [Besse 1987; Page 265].

1.0 Introduction

Harmonic maps

The theory of harmonic maps studies mappings between Riemannian manifolds from the energy minimization point of view. These were formally introduced by Eells-Sampson in [Eells and Sampson 1964], following the work of Fuller [Fuller 1954].

For a map $\varphi:(M^m,g)\to(N^n,h)$, the *energy* of φ over any compact domain $\Omega\subset M$ is given by

$$E(\varphi)=\frac{1}{2}\int_\Omega |d\varphi|^2 v_g$$

where $|d\varphi|^2 = g^{ij}\frac{\partial\varphi^\alpha}{\partial x^i}\frac{\partial\varphi^\beta}{\partial\varphi^j}h_{\alpha\beta}$.

A map $\varphi:(M^m,g)\to(N^n,h)$ is said to be *harmonic* if it is critical point of the energy $E(\varphi)$. That is, φ is a solution of the Euler Lagrange equations

$$\tau(\varphi)=0 \qquad (1)$$

or in local coordinates

$$\tau^\gamma(\varphi)=g^{ij}\left(\frac{\partial^2\varphi^\gamma}{\partial x^i\partial x^j}-\Gamma_{ij}^k\frac{\partial\varphi^\gamma}{\partial x^k}+\mathrm{L}_{\alpha\beta}^\gamma\frac{\partial\varphi^\alpha}{\partial x^i}\frac{\partial\varphi^\beta}{\partial x^j}\right)=0$$

where Γ_{ij}^k , $\mathrm{L}_{\alpha\beta}^\gamma$ are Christoffel symbols of M,N respectively. Many global existence results are known for this system of partial differential equations, cf. [Eells and Lemaire 1978, 1988].

These maps generalize the notions of geodesics and harmonic functions which have numerous physical applications ranging from theory of gravitation to the theory of electrostatics. Other important examples include minimal immersions, totally geodesic maps, Lie group homomorphisms and $\pm$ holomorphic maps between Kaehler manifolds.

The richness of the theory of harmonic maps has lead to highly interesting applications in various fields. For instance, these play a significant role in research related to

- minimal surfaces
- non-linear sigma models and chiral models in physics
- liquid crystals
- surface matching and image processing

Since the celebrated work of Eells and Sampson [Eells and Sampson 1964], the theory of harmonic mappings has grown substantially and has been intensely studied by many researchers. An online bibliography of harmonic maps is available at http://www.bath.ac.uk/~masfeb/harmonic.html.

Harmonic morphisms

A class of harmonic maps with a rich geometric character is the class of harmonic morphisms. These are maps between Riemannian manifolds which pull back germs of harmonic functions to harmonic functions. Precisely,

Definition 1.1:

A map $\varphi:(M^m,g)\to(N^n,h)$ *is said to be a harmonic morphism if for every open subset U of N (with* $\varphi^{-1}(U)$ *non-empty) and every harmonic function* $f:U\to R$, *the composition* $f\circ\varphi:\varphi^{-1}(U)\to R$ *is harmonic.*

The roots of the notion of harmonic morphisms lie in the work of Jacobi [Jacobi 1848] on the solutions of Laplace's equation. The preliminary work of the modern concept of harmonic morphisms was carried out by Constantinescu and Cornea in [Constantinescu and Cornea 1965], as the study of harmonic spaces in potential theory. However, the theory did not take off until the pioneering work of Fuglede [Fuglede 1978] and Ishihara [Ishihara 1979] where they independently introduced the formal theory of harmonic morphisms and established their fundamental properties. Their remarkable result, catching the interest of geometers, is the characterization of harmonic morphisms as a subclass of harmonic maps having strong geometric features. One says that a smooth map $\varphi:(M^m,g)\to(N^n,h)$ between Riemannian manifolds is *horizontally (weakly) conformal* if

$$g^{ij}\frac{\partial\varphi^\alpha}{\partial x^i}\frac{\partial\varphi^\beta}{\partial x^j}=\lambda^2(x)h^{\alpha\beta} \qquad (2)$$

for some function $\lambda:M\to[0,\infty)$ called as the *dilation* of φ.

<u>Theorem 1.2:</u> (Characterization) [Fuglede 1978, Ishihara 1979]

A smooth map $\varphi:(M^m,g)\to(N^n,h)$ between Riemannian manifolds is a harmonic morphism if and only if it is harmonic and horizontally (weakly) conformal.

Interesting examples of harmonic morphisms include

- weakly conformal maps between surfaces
- $\pm$ holomorphic maps from Kaehler manifolds to Riemann surfaces
- The natural projection from a Lie group G to the homogeneous space G/K
- Hopf fibrations
- Natural projections from warped products

Harmonic morphisms from a Riemannian manifold *M* to a surface have intriguing geometric properties. For instance, the regular fibres of such harmonic morphisms are minimal submanifolds of *M*. The subject of minimal submanifolds is a classical topic in Differential Geometry which in the last decades of 20th century has received astonishing new impetus, with the discovery of a huge variety of interesting new examples. The close relation of harmonic morphisms with minimal submanifolds makes these a particularly interesting topic of research.

Other variants of harmonic morphisms introduced recently are pluri-harmonic morphisms [Loubeau 1999] and biharmonic morphisms [Loubeau and Ou 2002]. An updated online bibliography of the rapidly growing area of harmonic morphisms is maintained by Gudmundsson at the following URL.
http://www.matematik.lu.se/matematiklu/personal/sigma/harmonic/bibliography.html

2.0 Literature Review

The characterization of harmonic morphisms as harmonic and horizontally (weakly) conformal maps implies that harmonic morphisms are solutions of an over determined system of non-linear partial differential equations. Thus one can not expect a general existence theory for harmonic morphisms. This makes the study of questions related to the existence, construction, classification and structure of harmonic morphisms rather difficult but of prime interest.

Until now a large amount of research, investigating the existence and classification of harmonic morphisms, has been carried out by considering harmonic morphisms between particular kind of manifolds or by considering special types of harmonic morphisms or by considering a combination of both. Such approach has lead to many existence and (partial or complete) classification results, and important constructions for (globally or locally) defined harmonic morphisms. For instance, the following have been particularly investigated and have provided significantly interesting results.

- Harmonic morphisms from 3-dimensional simply connected space forms to Riemann surfaces [Baird and Wood 1988, 1991].
- Polynomial harmonic morphisms [Ababou, Baird and Brossard 1999], [Baird 1983], [Baird and Ou 1997], [Ou 1997], [Ou and Wood 1996], [Eells and Yiu 1995].
- Complex valued harmonic morphisms from Euclidean spaces [Baird and Wood 1995, 1997].

- Horizontally homothetic harmonic morphisms, with totally geodesic fibres and integrable horizontal distribution, between simply connected space forms [Gudmundsson 1992, 1993].
- Holomorphic harmonic morphisms for $U \subset C^m$ to C^n [Gudmundsson and Sigurdsson 1993].
- Harmonic morphisms from various symmetric spaces [Gudmundsson 1994, 1995, 1997], [Gudmundsson and Svensson 2003a, 2003b].
- Harmonic morphisms with one-dimensional fibres [Bryant 2000], [Pantilie 1999, 2002], [Pantilie and Wood 2002b, 2003]
- Harmonic morphisms generated by foliations [Pantilie 2000a, 2000b].
- Harmonic morphisms from Einstein manifolds [Pantilie 2002,], [Pantilie and Wood 2002a, 2002b], [Ville 2003].

A natural tool for obtaining restrictions on existence of geometric objects is the use of a suitable Weitzenbock formula, commonly known as Bochner technique. For harmonic morphisms, it is developed in [Mustafa 1998, 1999, 2004] and several significant results regarding constraints on the existence of harmonic morphisms are obtained.

Among different types of harmonic morphisms, a class directly related to a geometric structure of physical interest consists of harmonic morphisms of warped product type.

<u>Definition 2.1:</u> [Baird and Wood 2003 ;Section 12.4 for details]

A map is called a harmonic morphism of warped product type if it is a non-constant horizontally homothetic harmonic morphism with totally geodesic fibres and integrable horizontal distribution. These maps are locally the projection of a warped product.

Harmonic morphisms of warped product type have been investigated in [Bryant 2000], [Gudmudsson 1992, 1993], [Pantilie 2002], [Pantilie and Wood 2002b]. In the context of Einstein manifolds, these have been studied in [Pantilie

2002], [Pantilie and Wood 2002b] where the constructions involving harmonic morphisms of warped product type are discussed. However, the results have not lead to any non-trivial example of harmonic morphisms of warped product type from compact Einstein manifolds; where a trivial harmonic morphism of warped product type means a map which is locally the projection of a Riemannian product. The only result known in this context, proved in the recent book by Baird and wood [Baird and Wood 2003; Proposition 12.7.1], is for one dimensional fibres.

> *Let* $\varphi:(M^{n+1},g)\to(N^{n},h)$ *be a harmonic morphism of warped product type from a compact manifold. If* M *is Einstein then up to a homothety* φ *is locally the projection of a Riemannian product.*

The aim of this project is to look into the general situation, i.e., to explore the constraints on the existence of harmonic morphisms of warped product type (with fibres of any dimension) from compact Einstein manifolds. Precisely, the following question will be looked at

> *Are there any harmonic morphisms of warped product type from compact Einstein manifolds? If yes, what are the restrictions on their existence?*

3.0 Objectives of the study

The primary aim of the work is to devise a method to obtain restrictions on the existence of harmonic morphisms of warped product type. A secondary goal is to draw significant applications. The main objectives to be achieved consist of following.

1. To develop a new Weitzenbock formula for harmonic morphisms of warped product type that only involves curvature of the domain manifold.

2. To analyze the Weitzenbock formula to obtain consequences about the existence of harmonic morphisms of warped product type from compact Einstein manifolds.
3. To extend the above method to obtain consequences for harmonic morphisms of warped product type from non-compact as well as semi-Riemannian manifolds.

4.0 Methodology

In order to achieve above mentioned objectives, following methodology will be followed;

4.1 Objective #1

The already existing Weitzenbock formulas for harmonic morphisms, developed by the principal investigator in [Mustafa 1998, 1999], involve curvatures of both the domain and target manifold, which were obtained by extension of a Weitzenbock formula for harmonic maps. The method here will be to calculate the Laplacian of a suitable object which extends to a new Weitzenbock formula that only involves (weakest possible form of) curvature of the domain manifold. Most likely, it can be done by exploiting the properties of the underlying warped product structure.

4.2 Objective #2

The first step would be to use the standard Bochner type argument, i.e. to impose suitable curvature condition (on the domain manifold) in the Weitzenbock formula followed by the application of a maximum principle to obtain required restrictions. The next step would be to attempt to improve the results obtained by the standard Bochner type argument. For this purpose a suitable adaptation of a maximum principle coupled with an estimation procedure will be considered. It can lead to obtaining non-existence results for harmonic morphisms of warped product type without imposing any curvature conditions.

4.3 Objective#3

The Weitzenbock formula can be extended by relating the Laplacian, calculated in objective#1, to the Laplacian of the fibres. Then the above procedure can be applied to investigate harmonic morphisms of warped product type (with compact Riemannian fibres) from non-compact as well as semi-Riemannian Einstein manifolds.

5.0 Management Plan

Both investigators involved in this research are faculty members in the department of mathematical sciences at KFUPM. In general, the investigators have research interests related to geometry and global analysis. The Principal Investigator (Tahir Mustafa) has a particular interest in the field of harmonic morphisms. He has extensively studied the Bochner technique which is one of the main tools needed for this work. The research work of the Co-Investigator (Hassan Azad) represents a broad spectrum of experience in the field of Differential Geometry, Algebraic Geometry and related areas. Jointly, the investigators provide the necessary capabilities to bring the proposed research project to a successful conclusion.

The stated objectives would be achieved through the collaborative work of the investigators. The principal Investigator will be coordinating the reports/publication outputs and perform necessary administrative management of the project.

6.0 Significance of the study

- Firstly: The project is concerned with understanding the structure of a class of harmonic morphisms; this is the fundamental issue in the research related to harmonic morphisms.

- Secondly: It also addresses the longstanding open question about the existence of compact Einstein warped products.

- Thirdly: Harmonic morphisms is an emerging field and many research groups in the world are turning towards this rapidly growing area. This joint project will play a significant role in the development of a group activity in the field of harmonic morphisms at KFUPM.

7.0 Scheduling of work

Task	**Months**											
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
Literature collection and completion of review												
Objective #1												
First Progress report												
Objective #2												
Objective #3												
Final Report and publication write up												

8.0 References

Ababou, R., Baird, P. and Brossard, J. (1999). Polynomes semi-conformes et morphisms harmoniques. *Math. Z.,* **231**, 589-604.

Baird, P. (1983). *Harmonic maps with symmetry, harmonic morphisms and deformation of metrics.* Research Notes in Mathematics, **87**, Pitman, Boston.

Baird, P. and Ou, Y. L. (1997). Harmonic maps and morphisms from multilinear norm-preserving mappings. *Internat. J. Math.,* **8**, 187-211.

Baird, P. and Wood, J. C. (1988). Bernstein theorems for harmonic morphisms from R^3 and S^3. *Math. Ann.,* **280**, 579-603.

Baird, P. and Wood, J. C. (1991). Harmonic morphisms and conformal foliations by geodesics of three-dimensional space forms. *J. Austral. Math. Soc.,* **51**, 118-153.

Baird, P. and Wood, J. C. (1995). Hermitian structures and harmonic morphisms in higher dimensional Euclidean spaces. *Internat. J. Math.,* **6**, 161-192.

Baird, P. and Wood, J. C. (1997). Weierstrauss representations for harmonic morphisms on Euclidean spaces and spheres. *Math. Scand.,* **81**, 283-300.

Baird, P. and Wood, J. C. (2003). *Harmonic morphisms between Riemannian manifolds.* London Math. Soc. Monogr., **29**, Oxford Univ. Press.

Besse, A. L. (1987). *Einstein manifolds.* Ergebnisse der Mathematik und ihrer Grenzgebiete, **10**, Springer, Berlin.

Bryant, R. L. (2000). Harmonic morphisms with fibres of dimension one. *Comm. Ann. Geom.,* **8**, 219-265.

Constantinescu, C. and Cornea, C. (1965). Compactifications of harmonic spaces. *Nagoya Math. J.,* **25**, 1-57.

Eells, J. and Lemaire, L. (1978). A report on harmonic maps. *Bull. London Math. Soc.,* **10**, 1-68.

Eells, J. and Lemaire, L. (1988). Another report on harmonic maps. *Bull. London Math. Soc.,* **20**, 385-524.

Eells, J. and Sampson, J. H. (1964). Harmonic mappings of Riemannian manifolds. *Amer J. Math.,* **86**, 109-160.

Eells, J. and Yiu, P. (1995). Polynomial harmonic morphisms between Euclidean spheres. *Proc. Amer. Math. Soc.,* **123**, 2921-2925.

Fuglede, B. (1978). Harmonic morphisms between Riemannian manifolds. *Ann. Inst. Fourier (Grenoble),* **28**, 107-144.

Fuller, F. B. (1954). Harmonic mappings. *Proc. Nat. Acad. Sci. U.S.A.,* **40**, 987-991.

Gudmundsson, S. (1992). *The geometry of harmonic morphisms*. Ph.D. Thesis. University of Leeds.

Gudmundsson, S. (1993). Harmonic morphisms between spaces of constant curvature. *Proc. Edinburgh Math. Soc.,* **36**, 133-143.

Gudmundsson, S. (1994). Harmonic morphisms from complex projective spaces. *Geom. Dedicata,* **53**, 155-161.

Gudmundsson, S. (1995). Harmonic morphisms from quaternionic projective spaces. *Geom. Dedicata,* **56**, 327-332.

Gudmundsson, S. (1997). On the existence of harmonic morphisms from symmetric spaces of rank one. *Manuscr. Math.,* **93**, 421-433.

Gudmundsson, S. and Sigurdsson, R. (1993). A note on the classification of holomorphic harmonic morphisms. *Potential Analysis,* **2**, 295-298.

Gudmundsson, S. and Svensson, M. (2003a). Harmonic morphisms from Grassmannians and their non-compact duals. Preprint, Lund University.

Gudmundsson, S. and Svensson, M. (2003b). Harmonic morphisms from the compact simple Lie groups and their non-compact duals. Preprint, Lund University.

Ishihara, T. (1979). A mapping of Riemannian manifolds which preserves harmonic functions. *J. Math. Kyoto Univ.,* **19**, 215-229.

Jacobi, C. G. J. (1848). Uber eine Losung der partiellen Differentialgleichung $\frac{\partial^2 V}{\partial^2 x^2}+\frac{\partial^2 V}{\partial^2 y^2}+\frac{\partial^2 V}{\partial^2 z^2}=0$. *J. Reine Angew. Math.,* **36**, 113-134.

Loubeau, E. (1999). Pluriharmonic morphisms. *Math. Scand.,* **84**, 165-178.

Loubeau, E. and Ou, Y. L. (2002). The characterization of biharmonic morphisms. *Differential geometry and its applications, Proc. Conf. Opava (Czech Republic), (August 27-31, 2001), Silesian University, Opava (to appear).*

Mustafa, M. T. (1998). A Bochner technique for harmonic morphisms. *J. London Math. Soc.,* **57**, 746-756.

Mustafa, M. T. (1999). Restrictions on harmonic morphisms. *Conformal Geom. Dyn.,* **3**, 102-115.

Mustafa, M. T. (2004). The structure of harmonic morphisms with totally geodesic fibres. *Communications in Contemporary Mathematics.,* (to appear).

Ou, Y. L. (1997). Quadratic harmonic morphisms and O-systems. *Ann. Inst. Fourier (Grenoble),* **47**, 687-713.

Ou, Y. L. and Wood, J. C. (1996). On the classification of harmonic morphisms between Euclidean spaces. *Algebras, Groups and Geometries,* **13**, 41-53.

Pantilie, R. (1999). Harmonic morphisms with one-dimensional fibres. *Internat. J. Math.,* **10**, 457-501.

Pantilie, R. (2000a). Conformal actions and harmonic morphisms. *Math. Proc. Cambridge Philo. Soc.,* **129**, 527-547.

Pantilie, R. (2000b). Isometric actions and harmonic morphisms. *Michigan Math. J.,* **47**, 453-467.

Pantilie, R. (2002). Harmonic morphisms with 1-dimensional fibres on 4-dimensional Einstein manifolds. *Comm. Anal. Geom.,* **10**, 779-814.

Pantilie, R. and Wood, J. C. (2002a). A new construction of Einstein self-dual metrics. *Asian J. Math.,* **6**, 337-348.

Pantilie, R. and Wood, J. C. (2002b). Harmonic morphisms with one-dimensional fibres on Einstein manifolds. *Trans. Amer. Math. Soc.,* **354**, 4229-4243.

Pantilie, R. and Wood, J. C. (2003). Twistorial harmonic morphisms with one-dimensional fibres on self-dual four-manifolds. Preprint, University of Leeds.

Ville, M. (2003). Harmonic morphisms from Einstein 4-manifolds to Riemann surfaces. *Internat. J. Math.,* **1**4, 327-337.

9.0 Budget

BUDGET REQUESTED

	Investigator (s) name (s)		Level of effort in months		Funds requested in SR
			Academic year		
(1)	(a) PI	Muhammad Tahir Mustafa	2004-05 (12 months)		14,400
	(b) Co-I	Hassan Azad	2004-05 (12 months)		12,000
			Total (1)		26,400

(2)	(a) Graduate Student		6,000
	(b) Secretary		2,000
		Total (2)	8,000

Equipment's, materials and supplies

(3)	(a) Printer		1,500
	(b) Stationary/Misc.		3,000
	(c) Allocation for books		2,500
		Total (3)	7,000

Conferences

(4)	Conference for PI	10,000
	Conference for CI	10,000
	Total (4)	20,000

Grand Total (1+2+3+4)	61,400

10.0 Suggested referees

List of Referees

1. Dr. S. Gudmundsson

   Mathematics
   Faculty of Science
   Box 118
   22100 Lund, Sweden
   Tel: [++46] - (0)46 - 222 85 61
   Fax: [++46] - (0)46 - 222 42 13
   E-mail: Sigmundur.Gudmundsson@math.lu.se
   URL: http://www.maths.lth.se/matematiklu/personal/sigma/

2. Dr. E. Loubeau
   Département de Mathématiques
   Université de Brest
   6, Avenue Victor Le Gorgeu
   B.P. 809
   29285 BREST Cedex
   FRANCE
   Email: loubeau@univ-brest.fr
   Tel: (0)2 98 01 73 82
   URL: http://maths2.univ-brest.fr/~loubeau/

3. Dr. R. Pantilie
   Department of Pure Mathematics
   University of Leeds
   Leeds LS2 9JT, UK
   E-Mail: r.pantilie@leeds.ac.uk

4. Dr. A. H. Bokhari
   Mathematical Sciences
   KFUPM, Dhahran 31261.
   Tel: 860 4182
   E-Mail: abokhari@kfupm.edu.sa