Polynomial solutions of certain differential equations arising in physics

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Abstract

Conditions for the existence of polynomial solutions of certain second-order differential equations have recently been investigated by several authors. In this paper a new algorithmic procedure is given to determine necessary and sufficient conditions for a differential equation with polynomial coefficients containing parameters to admit polynomial solutions and to compute these solutions. The effectiveness of this approach is illustrated by applying it to determine new solutions of several differential equations of current interest. A comparative analysis is given to demonstrate the advantage of this algorithmic procedure over existing software.

Key words: Linear differential equations, polynomial solutions

1 Introduction

In [3] an approach based on linear algebra for investigating polynomial solutions of differential equations of the form $\sum_{k=0}^{N} p_k(x)D^k y = 0$, where p_k is a polynomial of degree at most k, is given in detail. However many equations of interest in applications are not of this form; we may cite for instance [9], where the authors consider certain equations that arise in mathematical physics but that are not of the form considered in [3]. In [9] Ciftci et al specifically give conditions for the existence of polynomial solutions of second order linear differential equations using, in particular, the asymptotic iteration method (AIM) they introduced in their earlier work [8].

In this paper we consider more general linear differential equations. Using a linear algebraic approach suggested by ideas of [3], we provide necessary and sufficient conditions for the existence of polynomial solutions of linear differential equations of arbitrary order with polynomial coefficients containing parameters and of arbitrary degree as well as an algorithmic procedure for the verification of this condition and for constructing these solutions. This is discussed in detail in Section 2. Necessary and sufficient conditions for the existence of polynomial solutions of Heun's equation are also given in Section 2.

In Section 3, the algorithmic procedure of Section 2 is implemented to determine the conditions that guarantee the existence of polynomial solutions of any degree and to find them, depending of course on the available computational power.

To illustrate the efficiency of this algorithmic procedure, a Maple code is implemented in several examples, some of which appear in [9]. We point out that in [9], the authors state that finding polynomial solutions is a problem that needs to be investigated.

In the last section we describe the advantage of the algorithm given in this paper over existing software. Let us note that various algorithms addressing different aspects of this subject have been given, for example, in [1], [2], [5] and [7]. For classical references, the reader is referred to [4], [6], [12], [14] and [15].

2 Polynomial Solutions

Throughout, $\mathbb{P}$ is the space of all real polynomials and $\mathbb{P}_n$ is the subspace of polynomials with degree at most n. Let $L : \mathbb{P} \to \mathbb{P}$ be the linear operator given $Ly = \sum_{k=0}^{N} p_k(x)D^k y$, where D is the usual differential operator and $p_k(x) = \sum_{h\geq 0} p_{kh}x^h$ is a polynomial of degree d_k (with the convention that the zero polynomial has degree $-\infty$ and that $D^0 y = y$). Our objective is to find a necessary and sufficient condition for the equation $Ly = 0$ to have non-trivial polynomial solutions. Although this can be achieved, for each specific case, by comparing coefficients (see for example the determinantal necessary condition in the recent paper [9] on Heun's equations), or by using the Asymptotic Iteration Method in the case of second-order equations [8], we feel that a systematic approach that works for differential equations of all orders and that can easily be implemented in a computer algebra system is more desirable.

Assume first that for some i $(0 \leq i \leq N), d_i > i$. Let $m = \max_{0\leq i\leq N} (d_i - i)$ and put $y = D^m z$. In this way, the equation $Ly = 0$ is equivalent to $Hz = 0$ where H is the linear operator $\sum_{k=1}^{m+N} a_k(x)D^K$, and $Ly = 0$ has a polynomial solution of degree $n \geq 0$ if and only if $Hz = 0$ has a polynomial solution of degree $n + m$. Clearly, for each nonnegative integer $n, \mathbb{P}_n$ is H-invariant, and H has thus the advantage over L of being directly amenable to an eigenvalue analysis as demonstrated below. We note here that the case when $d_i \leq i$ for all i has been investigated in detail in [3].

Let a_k $(k \geq 1)$ be the sequence of polynomials defined by $a_k = 0$ if $k < m$ and $a_k = p_{k-m}$ if $k \geq m$. Put $a_k(x) = \sum_{h\geq 0} a_{kh}x^h$, where $a_{kh} = 0$ if $k < h$. Since, for each nonnegative integer n, $H(x^n)$ is a scalar multiple of x^n plus lower order terms, we see that the matrix representation of H, with respect to the standard basis $B_n = \{1, x, ..., x^n\}$ of $\mathbb{P}_n$ is upper triangular and its eigenvalues are the coefficients of x^n in $H(x^n)$. More specifically, the $(n+1)\times(n+1)$ matrix A_n of H operating on $\mathbb{P}_n$ has (i,j)-th entry $\sum_{k\geq 1} a_{k,\,k+i-j}(j-k)_k$, i.e.

$$A_n = \left[\sum_{k\geq 1} a_{k,\,k+i-j}(j-k)_k\right]_{1\leq i,\,j\leq n+1}$$

where $(j-k)_k = (j-1)(j-2)\cdots(j-k)$, and where each row and column has at most $(N+m+1)$ nonzero entries. Clearly, the first m columns of A_n are zero and A_{n+1} is obtained by A_n by adding one row and one column at the end. As diagonal entries of A_n, all the eigenvalues of the operator H are real and are given by $\lambda_n = n!\sum_{k=1}^{n}\frac{a_{kk}}{(n-k)!}$ for $n\geq 1$ (note that $\lambda_0 = \lambda_1 = \cdots = \lambda_{m-1} = 0$). Each eigenvalue λ_n has an eigenpolynomial $y_n(x) = y_{n0} + y_{n1}x + \cdots + y_{nn}x^n$ of degree at most n and whose vector representation $(y_{n0}, ..., y_{nn})^T$ in the standard basis B_n can be directly computed from the homogeneous upper triangular system $(A_n - \lambda_n I)(y_{n0}, ..., y_{nn})^T = 0$. Our problem is to find necessary and sufficient conditions for which the operator H has an eigenpolynomial of degree $n+m$ corresponding to $\lambda_{n+m} = 0$, that is necessary and sufficient conditions for the homogeneous system $A_{n+m}(y_{n+m,\,0}, ..., y_{n+m,n+m})^T = 0$ to have a solution $(y_{n+m,\,0}, ..., y_{n+m,n+m})^T$ with $y_{n+m,n+m} = 1$. This will follow from

Lemma 1. Let A be an $m\times n$ matrix. Then the homogeneous system $AX = 0$ has a solution $X = (x_1, x_2, ..., x_n)^T$ with $x_k \neq 0$ for some k if and only if $rank(A) = rank(A_k)$ *where* (A_k) *is the matrix obtained from* A *by deleting the* k^{th} *column.*

Proof. Put $A = [c_{ij}]_{1\leq i,j\leq n}$ and let c_k be the k^{th} column of A. Clearly, A and the augmented matrix $[A_k \vdots c_k]$ have the same rank. Hence,

$$rank(A) = rank(A_k) \Leftrightarrow rank[A_k \vdots c_k] = rank(A_k) \Leftrightarrow \text{ the system} A_k X = -c_k \text{ is consistent}$$
$$\Leftrightarrow \text{there exists a solution } X = (x_1, \cdots, x_{k-1}, 1, x_{k+1}, \cdots, x_n)^T \text{ to the system } AX = 0. \quad \square$$

Since A_{n+m-1} is obtained from A_{n+m} by deleting the last column, the above lemma immediately yields that the differential equation $Ly = 0$ has a polynomial solution of degree $n\geq 0$ if and only if $rank(A_{n+m}) = rank(A_{n+m-1})$. In this case, since A_{n+m} is upper triangular, the last entry of A_{n+m} is zero i.e. $\lambda_{n+m} = \sum_{k\geq 1} a_{kk}(m+n-k)_k = 0$, and therefore the last row of A_{n+m} is zero.

Now let M_n and M_n' be, respectively, the matrices obtained from A_{n+m} and A_{n+m-1} by deleting the first m zero columns. Clearly $rank(A_{n+m}) = rank(A_{n+m-1})$ if and only if

$rank(M_n) = rank(M'_n)$. It is easy to see that the $(i,j)^{th}$ entry of the $(n+m+1)\times(n+1)$ matrix M_n is $\sum_{t=0}^{j-1} a_{t+m,\,t+i-j}(j-t)_{t+m} = \sum_{t=0}^{j-1} p_{t,\,t+i-j}(j-t)_{t+m}$. This proves the main result of this note:

Proposition 2. With the above notation, let M_n be the $(n+m+1)\times(n+1)$ matrix with $(i,j)^{th}$ entry $\sum_{t=0}^{j-1} p_{t,\,t+i-j}(j-t)_{t+m}$ and let M'_n be the matrix obtained from M_n by deleting the last column. Then the differential equation $Ly = 0$ has a polynomial solution of degree $n \geq 0$ if and only if $rank(M_n) = rank(M'_n)$.$\square$

It thus follows that if the equation $Ly = 0$ has a polynomial solution of degree $n \geq 0$, then $\lambda_{n+m} = \sum_{t\geq 1} p_{t,\,t+m}(n-t)_{(t+m)} = 0$, and since M'_n has n columns, $rank(M_n) = rank(M'_n)$ implies that $rank(M_n) \leq n$ and so every $(n+1)\times(n+1)$ submatrix of M_n has zero determinant. This generalizes Theorems 5 and 6 of [9].

The following proposition, whose proof is implicit in the proof of Proposition 2, leads to an alternate algorithm for determining the conditions for existence of polynomial solutions of differential equations.

Proposition 3. Let L be the operator defined by $L(y) = \sum_{k=0}^{N} a_k(x)D^k(y)$. Let d_i be the degree of a_i and let

$$m = \max\{0, d_i : 0 \leq i \leq N\}.$$

Let H be the operator defined by by $H(y) = \sum_{k=0}^{N} a_k(x)D^{k+m}(y)$. A necessary condition for the equation $L(y) = 0$ to have a non-zero polynomial solution of degree at most n is that 0 must occur as an eigenvalue of the operator H with multiplicity at least $(m+1)$. Moreover the eigenvalues of H are the coefficients of x^n in $H(x^n)$ for $n = 0, 1, 2, \cdots$.

We conclude this section by proving a proposition on polynomial solutions, which may also be of independent interest and which can be extended to higher-order linear differential equations through appropriate modifications.

Let $y = \sum_{j\geq 0} c_j x^j$ be a solution of the differential equation

$$Ly \equiv (rx^3 + sx^2 + tx)y'' + (bx^2 + cx + \delta)y' + (\varepsilon x + f)y = 0 \tag{2.1}$$

Standard manipulations give $\alpha(k)c_{k-1} = \beta(k)c_k + \gamma(k)c_{k+1}$, for $k \geq 0$ (and $c_{-1} = 0$) where

$$\begin{aligned} \alpha(k) &= r(k-1)(k-2) + b(k-1) + \varepsilon \\ \beta(k) &= -sk(k-1) - ck - f \\ \gamma(k) &= -tk(k+1) - \delta(k+1) \end{aligned}$$

Suppose that y is monic of degree n, then $\alpha(n+1) = n(n-1) + bn + \varepsilon = 0$ (since $c_{n+1} = c_{n+2} = 0$). If n is the smallest positive integer for which there is a monic solution of (2.1) with degree n, then $\alpha(n+1) = 0$ and $\alpha(k) \neq 0$ for $1 \leq k \leq n$, so that $c_{k-1} = \dfrac{\beta(k)}{\alpha(k)}c_k + \dfrac{\gamma(k)}{\alpha(k)}c_{k+1}$.

For each k $(1 \leq k \leq n)$, let $A_k = \begin{bmatrix} \dfrac{\beta(k)}{\alpha(k)} & \dfrac{\gamma(k)}{\alpha(k)} \\ 1 & 0 \end{bmatrix}$. Then $\begin{bmatrix} c_{k-1} \\ c_k \end{bmatrix} = A_k \begin{bmatrix} c_k \\ c_{k+1} \end{bmatrix}$ so that $\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \left(\prod_{k=1}^{n} A_k \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Putting $x = 0$ in (1) gives $fc_0 + \delta c_1 = 0$, and this implies $\begin{bmatrix} f & \delta \end{bmatrix} \left(\prod_{k=1}^{n} A_k \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$. We have thus proved that if the DE (2.1) has a monic solution of smallest degree n then $\alpha(n+1) = 0$ and $\begin{bmatrix} f & \delta \end{bmatrix} \left(\prod_{k=1}^{n} A_k \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$. Conversely, assume there exists a positive integer n such that $\alpha(n+1) = 0$ and $\begin{bmatrix} f & \delta \end{bmatrix} \left(\prod_{k=1}^{n} A_k \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$. Define a sequence $(c_j)_{j \geq 0}$ by

$$\begin{aligned} c_j = \quad & 0 \text{ if } j > n \\ c_n = \quad & 1 \\ c_{j-1} = \quad & \frac{\beta(j)}{\alpha(j)}c_j + \frac{\gamma(j)}{\alpha(j)}c_{j+1} \text{ if } 0 \leq j \leq n-1 \end{aligned}$$

It is then easy to verify, by reversing the standard manipulations mentioned above, that the monic polynomial $\sum_{j \geq 0} c_j x^j$ of degree n is a solution of the DE (2.1). We summarize this in the following result.

Proposition 4. With the above notation, the DE (2.1) has a monic solution of degree n and no other monic solution of smaller degree iff $\alpha(n+1) = 0$ and $\begin{bmatrix} f & \delta \end{bmatrix} \left(\prod_{k=1}^{n} A_k \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$. This monic solution is given by $y = x^n + \sum_{k=0}^{n-1} \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\prod_{j=k+1}^{n} A_j \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} x^k$.

Note that the coefficients of the monic solution above can also be obtained from the following general result, which can easily be proved by induction.

Lemma. Let a sequence $(c_k)_{1 \leq k \leq n+1}$ be given by $c_n = 1$, $c_{n+1} = 0$ and $c_{k-1} = u_k c_k + v_k c_{k+1}$. Then c_k $(0 \leq k \leq n-1)$ is the principal $(n-k) \times (n-k)$ minor of the $n \times n$

tridiagonal matrix

$$\begin{bmatrix} u_n & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ v_{n-1} & u_{n-1} & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & v_{n-2} & u_{n-2} & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & v_{n-3} & u_{n-3} & -1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & v_2 & u_2 & -1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & v_1 & u_1 \end{bmatrix}$$

3 Examples

In this section we illustrate the effectiveness of the algorithmic approach of Section 2 by applying it to different types of differential equations. Examples include the general Heun's equation as well as some physically significant differential equations that arise in the study of solutions to Schrödinger equation [9, 13] and radial Schrödinger equation with shifted potential [9, 10, 11].

These examples show how to implement the method algorithmically to determine the conditions for the existence of polynomial solutions and also to calculate the corresponding polynomial solutions. As a test case, the procedure is also applied to obtain polynomial solutions of a third order differential equation. All the solutions presented here have been verified using Maple.

In case of examples where the implementation of Proposition 2 involves a higher complexity, the alternate approach based on Proposition 3 can be adopted to obtain polynomial solutions as well as conditions for their existence. This approach is illustrated in the following example.

Example 1

We consider the general Heun's differential equation of the form

$$L(y) = \left(x^3 - (a+1)\,x^2 + ax\right)\frac{d^2y}{dx^2} + \left(bx^2 + cx + \delta\right)\frac{dy}{dx} + (\epsilon\,x + f)\,y = 0 \tag{3.1}$$

Since $m = 1$, in the notation of Proposition 3, the substitution $y = \dfrac{dz}{dx}$ leads to

$$H(z) = \left(x^3 - (a+1)\,x^2 + ax\right)\frac{d^3z}{dx^3} + \left(bx^2 + cx + \delta\right)\frac{d^2z}{dx^2} + (\epsilon\,x + f)\,\frac{dz}{dx} = 0 \tag{3.2}$$

Using

$$\begin{aligned} H(x^n) &= n(n^2 - 3n + 2 + bn - b + \epsilon)x^n + n(-an^2 + 3an - 2a - n^2 + 3n - 2 + cn - c + f)x^{n-1} \\ &\quad + n(an^2 - 3an + 2a + \delta n - \delta)x^{n-2} \end{aligned}$$

we see that the matrix A of H relative to the basis $\{1, x, x^2, \cdots, x^n\}$ is upper triangular and has

$$A(n+1, n+1) = n^2 - 3n + 2 + b(n-1) + \epsilon.$$

So by Proposition 3 or Proposition 4, a necessary condition for ODE (3.2) to have polynomial solution of degree $n \geq 2$ is obtained using $A(n+1, n+1) = 0$ as

$$b = -\frac{n^2 - 3n + 2 + \epsilon}{n-1}.$$

Further if two rows of A are linearly dependent then the rank of the matrix of H on $\mathbb{P}_n$ equals the rank of matrix of H on $\mathbb{P}_{n-1}$, which ensures the existence of polynomial solutions of ODE (3.2) of degree $n \geq 2$ and hence of degree $(n-1)$ for ODE (3.1). This approach can be implemented easily in many ways as illustrated in the cases below.

Case (1): For any $n \geq 2$, the choice of parameters

$$a \neq 0,\ b = -\frac{n^2 - 3n + 2 + \epsilon}{n-1},\ c \neq -f,\ \delta = -a,\ \epsilon = -\frac{(c+f)f}{a},\ f \neq 0$$

gives $A(n+1, n+1) = 0$, and makes rows 1 and 2 of A linearly dependent. Hence these parameters ensure the existence of polynomial solutions of degree $(n-1)$ of ODE (3.1) which can directly be computed via null space of A. Some examples of solutions for these parameters are given in Table 1 below.

n	Polynomial solution of ODE (3.1) of degree $(n-1)$
2	$2\,x + 2\,\frac{a}{f}$
3	$3\,x^2 - 6\,\frac{(2\,a-2\,c-f+2)ax}{cf+f^2+2\,a} - 6\,\frac{a^2(2\,a-2\,c-f+2)}{(cf+f^2+2\,a)f}$
4	$4\,x^3 - 12\,\frac{(6\,a+6-f-3\,c)ax^2}{6\,a+fc+f^2} + 36\,\frac{a^2\left(-9\,ca-9\,c+3\,fc+3\,c^2+6\,a^2+15\,a-4\,af+6-4\,f+f^2\right)x}{(3\,a+fc+f^2)(6\,a+fc+f^2)}$ $+36\,\frac{a^3\left(-9\,ca-9\,c+3\,fc+3\,c^2+6\,a^2+15\,a-4\,af+6-4\,f+f^2\right)}{(3\,a+fc+f^2)(6\,a+fc+f^2)f}$
5	$5\,x^4 - 20\,\frac{(12\,a+12-f-4\,c)ax^3}{12\,a+fc+f^2} + 120\,\frac{a^2\left(24\,a^2+56\,a-6\,af-20\,ac+24-6\,f-20\,c+f^2+3\,fc+4\,c^2\right)x^2}{(12\,a+fc+f^2)(8\,a+fc+f^2)}$ $-80\,\frac{a^3Ax}{(4\,a+fc+f^2)(8\,a+fc+f^2)(12\,a+fc+f^2)}$ $-80\,\frac{a^4A}{(4\,a+fc+f^2)(8\,a+fc+f^2)(12\,a+fc+f^2)f}$ where $A = 96 + 416a - 176c - 72f + 88fc + 28f^2 + 88caf + 416a^2 - 168af - 416ac + 96c^2 - 176ca^2$ $+96ac^2 - 15cf^2 - 24fc^2 - 72a^2f + 28af^2 - 16c^3 + 96a^3 - 3f^3$

Table 1: Polynomial solutions of ODE (3.1)

Higher degree solutions are obtained very efficiently but cannot be reproduced here, for general parameters, due to space constraint. For example a polynomial solution of degree 25 of ODE (3.1) for a specific choice of parameters was computed without difficulty and is provided in [16].

Case (2): For any $n \geq 2$, the choice of parameters

$$a = -\lambda(b+\epsilon),\ b = -\frac{n^2-3n+2+\epsilon}{n-1},\ c = a+1,\ \delta = -2a,\ f = 0$$

gives $A(n+1, n+1) = 0$, and makes rows 1 and 3 of A linearly dependent. Hence the existence of polynomial solutions of degree $(n-1)$ of ODE (3.1) is ensured. As an example, the solution of degree 5 of ODE (3.1) with the above parameters is given below.

$$\begin{aligned}
& y(x) = 6\,x^5 - 90\,\frac{(4\,\lambda\,\epsilon - 20\,\lambda - 5)\,x^4}{\epsilon - 20} \\
+ \quad & 120\,\frac{(48\,\lambda^2\epsilon^2 - 480\,\lambda^2\epsilon + \lambda\,\epsilon^2 + 1200\,\lambda^2 - 145\,\lambda\,\epsilon + 700\,\lambda + 75)\,x^3}{(\epsilon - 20)\,(\epsilon - 15)} \\
- \quad & 120\frac{(4\lambda\epsilon - 20\lambda - 5)\,(48\lambda^2\epsilon^2 - 480\lambda^2\epsilon + 5\lambda\epsilon^2 + 1200\,\lambda^2 - 225\,\lambda\,\epsilon + 1000\,\lambda + 75)\,x^2}{(\epsilon - 20)\,(\epsilon - 15)\,(\epsilon - 10)} \\
+ \quad & 192\,\frac{\lambda\,(\epsilon - 5)\,(4\lambda\epsilon - 20\lambda - 5)\,(48\,\lambda^2\epsilon^2 - 480\lambda^2\epsilon + 5\lambda\epsilon^2 + 1200\lambda^2 - 225\lambda\epsilon + 1000\lambda + 75)}{(\epsilon - 20)\,(\epsilon - 15)\,(\epsilon - 10)\,\epsilon}
\end{aligned}$$

Case (3): For any $n \geq 2$, the choice of parameters

$$a = -\frac{2c^2 - 2f - 2c + 3cf + f^2}{2(-c - f + b)},\ b = -\frac{n^2 - 3n + 2 + \epsilon}{n-1},\ \delta = -2a,\ \epsilon = 0$$

gives $A(n+1, n+1) = 0$, and makes rows 2 and 3 of A linearly dependent. Hence the existence of polynomial solutions of degree $(n-1)$ of ODE (3.1) is ensured for these parameters. An example of the solution of degree 5 of ODE (3.1) with the above parameters is given below.

$$\begin{aligned}
y(x) &= 6\,x^5 - \frac{3}{2}\,\frac{(15\,c^2 + 24\,cf + 9\,f^2 - 20\,c - 4\,f + 80)\,x^4}{c + f + 4} + \frac{3}{4}\,\frac{A_3 x^3}{(c + f + 4)^2} \\
&- \frac{1}{8}\,\frac{A_2 x^2}{(c + f + 4)^3} - \frac{1}{8}\,\frac{A_2\,(2\,c + f - 2)\,x}{(c + f + 4)^4} \\
&- \frac{1}{8}\,\frac{A_2\,(2\,c^2 + 3\,cf + f^2 - 2\,c - 2\,f)\,(2\,c + f - 2)}{(c + f + 4)^5\,f}
\end{aligned}$$

where

$$\begin{aligned}
A_3 &= 40\,c^4 + 129\,c^3 f + 153\,c^2 f^2 + 79\,cf^3 + 15\,f^4 - 120\,c^3 - 212\,c^2 f - 104\,cf^2 \\
&- 12\,f^3 + 600\,c^2 + 832\,cf + 296\,f^2 - 800\,c - 224\,f + 1280
\end{aligned}$$

and

$$\begin{aligned}
A_2 &= 120\,c^6 + 587\,c^5 f + 1184\,c^4 f^2 + 1260\,c^3 f^3 + 746\,c^2 f^4 + 233\,cf^5 + 30\,f^6 - 720\,c^5 \\
&- 2452\,c^4 f - 3172\,c^3 f^2 - 1916\,c^2 f^3 - 524\,cf^4 - 48\,f^5 + 4400\,c^4 + 12240\,c^3 f \\
&+ 12720\,c^2 f^2 + 5936\,cf^3 + 1056\,f^4 - 12800\,c^3 - 22528\,c^2 f - 12160\,cf^2 - 1920\,f^3 \\
&+ 30400\,c^2 + 35584\,cf + 11328\,f^2 - 37120\,c - 13056\,f + 30720.
\end{aligned}$$

Other choices of parameters leading to the existence of polynomials solutions are also possible. In general, the following procedure can be implemented iteratively. Let A^* denote the matrix obtained after making the entries $A(1,j)$ of A zero by using the entries $A(j,j)$ for $j = 2, 3, \cdots, n$. Then the parameters obtained by solving $A(n+1, n+1) = 0$ and $A^*(1, n+1) = 0$ will ensure the existence of a polynomial solution of degree $(n-1)$ of ODE (3.1).

Example 2

As a second example, we consider the linear second order ODE arising in the study of one dimensional Schrödinger problems [13]. The investigation of Krylov and Robnik [13] about polynomial solutions of one dimensional Schrödinger problems leads to investigation of polynomial solutions of the following differential equation

$$x^3 \frac{d^2y}{dx^2} + a\left(x^2 - 1\right)\frac{dy}{dx} + (\epsilon\, x + f)\, y = 0 \tag{3.3}$$

The conditions for the existence of polynomial solutions of this ODE have also been discussed by Ciftci et al. in [9], by a different approach. Here, we apply our method to determine existence conditions as well as to compute the corresponding polynomial solutions of the above differential equation. It is found that the best approach to deal with this ODE is to utilize Proposition 4.

For $f = 0$, in the notation of Proposition 4, we have

$$\begin{aligned} \alpha(k) &= k^2 + (a-3)k + \varepsilon - a + 2 \\ \beta(k) &= 0 \\ \gamma(k) &= (k+1)a \end{aligned}$$

and $A_k = \begin{bmatrix} 0 & a_k \\ 1 & 0 \end{bmatrix}$ where $a_k = \dfrac{(k+1)a}{(k^2 + (a-3)k + \varepsilon - a + 2)}$.
So, for $n = 2m$, we have

$$\left(\prod_{k=1}^{n} A_k\right) = \begin{bmatrix} a_1 a_3 \cdots a_m & 0 \\ 0 & a_2 a_4 \cdots a_{2m} \end{bmatrix},$$

implying that $\begin{bmatrix} 0 & -a \end{bmatrix} \left(\prod_{k=1}^{n} A_k\right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$. Hence, by Proposition 4, ODE (3.3) has polynomial solutions of even degree n for $f = 0$ iff $\epsilon = -n^2 + n - an$ and the solution expression of Proposition 4 can be implemented directly to generate a sequence of even degree polynomial solutions of ODE (3.3). A further analysis of these solutions yields the following result.

- For $f = 0$, ODE (3.3) admits polynomial solutions of degree $n = 2m$ $(m \geq 1)$, with $\epsilon = -(an + n^2 - n)$, given by

$$y = x^{2m} + \sum_{i=1}^{m} \frac{(-1)^i \binom{m}{i} a^i x^{2m-2i}}{(a+2n-3)(a+2n-5)\cdots(a+2n-3-2(i-1))}$$

For $n = 2m-1$ $(m \geq 1)$, we have

$$\left(\prod_{k=1}^{n} A_k\right) = \begin{bmatrix} 0 & a_1 a_3 \cdots a_m \\ a_2 a_4 \cdots a_{2m-2} & 0 \end{bmatrix},$$

implying that $\begin{bmatrix} 0 & -a \end{bmatrix} \left(\prod_{k=1}^{n} A_k\right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -a_2 a_4 \cdots a_{2m-2}$. Hence, by Proposition 4, ODE (3.3) has polynomial solutions of odd degree n for $f = 0$ iff $a = 0$ and $\epsilon = -n^2 + n$, which are given by $y(x) = x^n$.

For $f \neq 0$, there does not seem to be a simple pattern. Nevertheless for any given n and a, the algorithmic procedure of Proposition 4 can be easily implemented to determine ϵ, f for which ODE (3.3) admits polynomial solutions of degree n as well as to compute the corresponding polynomial solution.

As an illustration if we take $a = \frac{-15}{2}$ and look for a solution of degree $n = 6$ then the implementation of Proposition 4 determines $\epsilon = 15$, $f = 3(750)^{1/4}$ and computes the corresponding polynomial solution of degree 6 of ODE (3.3) as

$$\begin{aligned} y(x) = {} & x^6 + 6/5\, 5^{3/4} \sqrt[4]{6} x^5 + \left(6\sqrt{5}\sqrt{6} + 15\right) x^4 + \left(60\, 5^{3/4} \sqrt[4]{6} + 60 \sqrt[4]{5} 6^{3/4}\right) x^3 \\ & + \left(-2925 - 540\sqrt{5}\sqrt{6}\right) x^2 + \left(900 \sqrt[4]{5} 6^{3/4} + 990\, 5^{3/4} \sqrt[4]{6}\right) x - 450\sqrt{5}\sqrt{6} - 2475 \end{aligned}$$

Some other lower degree solutions are displayed in Table 2 below.

n	ϵ	f	Polynomial solution of ODE (3.3) of degree n
1	$-a$	$\pm a$	$x \pm 1$
2	$-2a-2$	$\pm\sqrt{4a^2+6a}$	$x^2 \pm \frac{\sqrt{2a(2a+3)}}{a+2} x + \frac{a}{a+2}$
3	$-3a-6$	$\pm\sqrt{15a+5a^2+aA}$	$x^3 \pm \frac{a\sqrt{(5a+15+A)a}(-9-2a+A)}{6(a+2)(a+3)(a+4)} + \frac{a(3+2a+A)x}{2(a+3)(a+4)} \pm \frac{\sqrt{(5a+15+A)a}x^2}{a+4}$
		$\pm\sqrt{15a+5a^2-aA}$	$x^3 \mp \frac{a\sqrt{(5a+15-A)a}(9+2a+A)}{6(a+2)(a+3)(a+4)} - \frac{a(-3-2a+A)x}{2(a+3)(a+4)} \pm \frac{\sqrt{(5a+15-A)a}x^2}{a+4}$ where $A = \sqrt{16a^2+96a+153}$

Table 2: Polynomial solutions of ODE (3.3)

Example 3

In this example, we consider a differential equation related to the investigation of the radial Schrödinger equation with shifted Coulomb potential and which has been discussed recently in [9, 10, 11]. The anstaz of [9, Eq.35] that the radial Schrödinger equation admits a solution which vanishes at the origin and at infinity leads to the question of obtaining solutions of the following differential equation:

$$
\begin{aligned}
L(y) &= x(x+\beta)\frac{d^2y}{dx^2} + \left(-2\alpha x^2 + 2(K+1-\alpha\beta)x + 2\beta(K+1)\right)\frac{dy}{dx} \\
&\quad + \left((-2\alpha(K+1)+2Z)x - 2\alpha\beta(K+1)\right)y = 0
\end{aligned}
\tag{3.4}
$$

This is a particular case of the confluent Heun equation whose polynomial solutions can be studied algorithmically using our procedure. While discussing the question of polynomial solutions of ODE (3.4), Ciftci et al. [9] give conditions on parameters α, β to have polynomial solutions. However, they state that finding the corresponding polynomial solutions is an open problem that remains to be solved. Adapting the same approach, as in Example 1, gives the existence conditions on parameters as well as generate the corresponding polynomial solutions of ODE (3.4). For instance for $n \geq 2$, as in Example 1, the choice of parameters

$$\alpha = \frac{Z}{n+K}, \quad \beta = \frac{Z}{\alpha^2(2+K)}, \quad K = -\frac{3}{2}$$

gives $A(n+1, n+1) = 0$, and makes rows 1 and 2 of A linearly dependent, where A is the matrix of operator $H(z)$ obtained by putting $y = \dfrac{dz}{dx}$ in ODE (3.4). Thus the existence of polynomial solutions of degree $(n-1)$ of ODE (3.4) is ensured. Explicit examples of such solutions are given in Table 3 below.

n	Polynomial solution of ODE (3.4) of degree $(n-1)$
2	$2\,x + \frac{1}{Z}$
3	$3\,x^2 + \frac{81}{4}\,\frac{x}{Z} + \frac{243}{8Z^2}$
4	$4\,x^3 + 110\,\frac{x^2}{Z} + \frac{1875}{2}\,\frac{x}{Z^2} + \frac{9375}{4Z^3}$
5	$5\,x^4 + \frac{1435}{4}\,\frac{x^3}{Z} + \frac{18375}{2}\,\frac{x^2}{Z^2} + \frac{3109295}{32}\,\frac{x}{Z^3} + \frac{21765065}{64Z^4}$
6	$6\,x^5 + 891\,\frac{x^4}{Z} + 51030\,\frac{x^3}{Z^2} + 1390932\,\frac{x^2}{Z^3} + \frac{282195171}{16}\,\frac{x}{Z^4} + \frac{2539756539}{32Z^5}$
7	$7\,x^6 + \frac{7469}{4}\,\frac{x^5}{Z} + \frac{1613535}{8}\,\frac{x^4}{Z^2} + \frac{22407385}{2}\,\frac{x^3}{Z^3} + \frac{10669409135}{32}\,\frac{x^2}{Z^4} + \frac{631442801913}{128}\,\frac{x}{Z^5} + \frac{6945870821043}{256Z^6}$
8	$8\,x^7 + 3484\,\frac{x^6}{Z} + 635271\,\frac{x^5}{Z^2} + \frac{125086195}{2}\,\frac{x^4}{Z^3} + \frac{14254081075}{4}\,\frac{x^3}{Z^4} + \frac{929027438313}{8}\,\frac{x^2}{Z^5}$ $+\frac{62844710476561}{32}\,\frac{x}{Z^6} + \frac{816981236195293}{64Z^7}$
9	$9\,x^8 + \frac{23895}{4}\,\frac{x^7}{Z} + 1701000\,\frac{x^6}{Z^2} + \frac{8652774375}{32}\,\frac{x^5}{Z^3} + \frac{1671312234375}{64}\,\frac{x^4}{Z^4} + \frac{199371259828125}{128}\,\frac{x^3}{Z^5}$ $+\frac{3545740214296875}{64}\,\frac{x^2}{Z^6} + \frac{1079266367548828125}{1024}\,\frac{x}{Z^7} + \frac{16188995513232421875}{2048Z^8}$
10	$10\,x^9 + 9605\,\frac{x^8}{Z} + 4031550\,\frac{x^7}{Z^2} + 967934695\,\frac{x^6}{Z^3} + \frac{291940556215}{2}\,\frac{x^5}{Z^4} + \frac{57076142912355}{4}\,\frac{x^4}{Z^5}$ $+\frac{7182343532112455}{8}\,\frac{x^3}{Z^6} + \frac{554919011624111525}{16}\,\frac{x^2}{Z^7} + \frac{187627110879149608965}{256}\,\frac{x}{Z^8} + \frac{318966088494554335240 5}{512Z^9}$
11	$11\,x^{10} + \frac{58729}{4}\,\frac{x^9}{Z} + \frac{69512355}{8}\,\frac{x^8}{Z^2} + \frac{11984847303}{4}\,\frac{x^7}{Z^3} + \frac{5319192752643}{8}\,\frac{x^6}{Z^4} + \frac{1581982015774545}{16}\,\frac{x^5}{Z^5}$ $+\frac{317780871715555941}{32}\,\frac{x^4}{Z^6} + \frac{42283382187962949825}{64}\,\frac{x^3}{Z^7} + \frac{14118520017945812674887}{512}\,\frac{x^2}{Z^8}$ $+\frac{1312232950923162294745169}{2048}\,\frac{x}{Z^9} + \frac{24932426067540083600158211}{4096Z^{10}}$

Table 3: Polynomial solution of ODE (3.4)

In general for a given K and a given degree of a polynomial solution, applying Propositions 2 or 3 can determine conditions on parameters α, β for the existence of polynomial solutions of ODE (3.4). We demonstrate this below by presenting a general implementation of Proposition 3 to obtain a polynomial solution.

If A denotes the matrix of H on $\mathbb{P}_n$ then as above $A(n+1,n+1)=0$ implies $\alpha=\frac{Z}{n+K}$. Fixing the degree of the required polynomial solution as 7 (i.e. $n=8$ in the above terminology), $K=-3$ and $\alpha=\frac{Z}{n+K}$ gives A as

$$\begin{bmatrix} 0 & \frac{4}{5}Z\beta & -8\,\beta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{14}{5}Z & -8+\frac{4}{5}Z\beta & -18\beta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{24}{5}Z & -18 & -24\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6Z & -24-\frac{8}{5}Z\beta & -20\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{32}{5}Z & -20-4Z\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6Z & -\frac{36}{5}Z\beta & 42\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{24}{5}Z & 42-\frac{56}{5}Z\beta & 112\beta \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{14}{5}Z & 112-16Z\beta \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Making the entries $A(1,j)$ zero using $A(j,j)$ for $j=2,3,\cdots,8$ gives

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A(1,9) \\ 0 & \frac{14}{5}Z & -8+\frac{4}{5}Z\beta & -18\beta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{24}{5}Z & -18 & -24\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6Z & -24-\frac{8}{5}Z\beta & -20\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{32}{5}Z & -20-4Z\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6Z & -\frac{36}{5}Z\beta & 42\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{24}{5}Z & 42-\frac{56}{5}Z\beta & 112\beta \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{14}{5}Z & 112-16Z\beta \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $A(1,9)=\frac{100}{7}\frac{\beta^2\left(Z^2\beta^2-50Z\beta-75\right)\left(-60Z\beta+175+4Z^2\beta^2\right)}{Z^5}$. This leads to the condition

$$\left(Z^2\beta^2-50Z\beta-75\right)\left(-60Z\beta+175+4Z^2\beta^2\right)=0$$

giving

$$\beta=5\,\frac{5\pm2\sqrt{7}}{Z},\ 5\,\frac{3\pm\sqrt{2}}{2Z}.$$

So for $K=-3$, $\alpha=\frac{Z}{n+K}=\frac{Z}{5}$ and

$$\beta=5\,\frac{5\pm2\sqrt{7}}{Z},\ 5\,\frac{3\pm\sqrt{2}}{2Z}$$

ODE (3.4) has a polynomial solution of degree 7 which can be computed directly through the null space of A. For instance, for $\beta = 5\,\frac{5+2\sqrt{7}}{Z}$ we get the solution

$$\begin{aligned}
y(x) &= 8\,x^7 + 80\,\frac{\left(9+5\sqrt{7}\right)x^6}{Z} + 300\,\frac{\left(277+100\sqrt{7}\right)x^5}{Z^2} + 2500\,\frac{\left(5+2\sqrt{7}\right)\left(159+56\sqrt{7}\right)x^4}{Z^3} \\
&+ 25000\,\frac{\left(5+2\sqrt{7}\right)^2\left(159+56\sqrt{7}\right)x^3}{\left(1+\sqrt{7}\right)Z^4} + 150000\,\frac{\left(5+2\sqrt{7}\right)^3\left(159+56\sqrt{7}\right)x^2}{\left(1+\sqrt{7}\right)^2 Z^5} \\
&+ 1125000\,\frac{\left(5+2\sqrt{7}\right)^4\left(159+56\sqrt{7}\right)x}{\left(1+\sqrt{7}\right)^2\left(5+\sqrt{7}\right)Z^6} + 5625000\,\frac{\left(5+2\sqrt{7}\right)^4\left(159+56\sqrt{7}\right)}{\left(5+\sqrt{7}\right)\left(1+\sqrt{7}\right)^2 Z^7}
\end{aligned}$$

Example 4

As a test case for an equation of defect 3, we consider a third order differential equation of the form

$$\left(1-x^2\right)^3\frac{d^3y}{dx^3} + ax\left(1-x^2\right)^2\frac{d^2y}{dx^2} + a\left(1-x^2\right)\left(bx^2-1\right)\frac{dy}{dx} + \left(cx^3+\delta\,x\right)y = 0 \qquad (3.5)$$

and implement our procedure to determine the values of parameters for the existence of polynomial solutions and then finding the corresponding solutions. This leads to the following ansätze for polynomial solutions of degree $2p$ of ODE (3.5):

$$c = 2\,p\left(4\,p^2 - 6\,p + 2 - 2\,ap + a + ab\right)$$

and

$$\delta = -c \qquad \text{or} \qquad \delta = -12p(p-1).$$

As an example, a 10^{th} degree solution of ODE (3.5) for $\delta = -c$ is computed as

$$\begin{aligned}
y(x) &= 1716\,x^{10} + 68640\,\frac{(a-18)\,x^8}{ab-17\,a+192} + 102960\,\frac{\left(8\,a^2+3\,ab-307\,a+2592\right)x^6}{(ab-17\,a+192)\,(ab-15\,a+150)} \\
&+ 137280\,\frac{\left(24\,a^3+37\,a^2b-1581\,a^2-594\,ab+28746\,a-153360\right)x^4}{(ab-17\,a+192)\,(ab-15\,a+150)\,(ab-13\,a+116)} \\
&+ 34320\frac{Ax^2}{(ab-17a+192)\,(ab-15a+150)\,(ab-13a+116)\,(ab-11a+90)} \\
&+ 329472\,\frac{24\,a^3+37\,a^2b-1581\,a^2-594\,ab+28746\,a-153360}{(ab-17\,a+192)\,(ab-15\,a+150)\,(ab-13\,a+116)\,(ab-9\,a+72)}
\end{aligned}$$

where

$$A = 96a^4+508a^3b+135a^2b^2-11580a^3-18834a^2b+374283a^2+146556ab-4422204a+17210880.$$

For $\delta = -12p(p-1)$, an example of solution of ODE (3.5) of degree $2p = 20$ is obtained as

$$\begin{aligned}
y(x) &= 10626\,x^{20} - 106260\,x^{18} + 478170\,x^{16} - 1275120\,x^{14} + 2231460\,x^{12} - 2677752\,x^{10} \\
&+ 2231460\,x^8 - 1275120\,x^6 + 478170\,x^4 - 106260\,x^2 + 10626.
\end{aligned}$$

In general, for a given degree $2p$, this procedure can be implemented to obtain the corresponding polynomial solutions of ODE (3.5)- if any exists - for the above choices of parameters $c,\ \delta$.

4 Comparison with other software packages

Maple has two available options that can be used to find polynomial solutions of differential equations with polynomial coefficients, namely the command *polysols* in the package DEtools and the command *PolynomialSolution* in the package LinearFunctionalSystems. Both these commands can only find polynomial solutions if their existence has already been determined but are unable to handle the existence question of polynomial solutions of differential equations with polynomial coefficients. Hence these are not suitable to study polynomial solutions of differential equations with arbitrary parameters, where the polynomial solutions may exist for some values of parameters; this deficiency is highlighted below.

Using the Maple commands *polysols* or *PolynomialSolution* for the ODE (3.3)

$$x^3 \frac{d^2y}{dx^2} + a\left(x^2 - 1\right)\frac{dy}{dx} + (\epsilon\, x + f)\, y = 0$$

returns no polynomial solution. However, after fixing those values of a, ϵ, f for which there is a solution of a given degree n, the Maple commands *polysols* or *PolynomialSolution* can find the corresponding polynomial solution.

The algorithmic procedure we presented here improves this deficiency in available Maple options as summarized below.
Given an ODE (with arbitrary parameters) having polynomial coefficients of any degree, the algorithm procedure of this paper

- Takes input as n which is the degree of desired polynomial solution.
- For this n,
  - it first finds the conditions on parameters for which polynomial solutions exist
  - and then it determines the corresponding polynomial solutions.

Hence it significantly improves upon the existing Maple commands for finding polynomial solutions of ODEs (with arbitrary parameters) having polynomial coefficients. In case of ODEs without arbitrary parameters it is as good as the existing Maple options. To our knowledge, there are no options available in Mathematica or Matlab that can treat the question of finding polynomial solutions of differential equations (with arbitrary parameters) having polynomial coefficients in as general manner as our algorithm does.

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