HARMONIC MORPHISMS FROM THREE-DIMENSIONAL EUCLIDEAN AND SPHERICAL SPACE FORMS

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Abstract

This paper gives a description of all harmonic morphisms from a three-dimensional *non-simply-connected* Euclidean and spherical space form to a surface, by extending the work of Baird-Wood [4, 5] who dealt with the simply-connected case; namely we show that any such harmonic morphism is the composition of a "standard" harmonic morphism and a weakly conformal map. To complete the description we list the space forms and the standard harmonic morphisms on them.

1. Introduction

A smooth map $\phi \colon M \to N$ between Riemannian manifolds is called a *harmonic morphism* if it preserves germs of harmonic functions, i.e., if f is a real-valued harmonic function on an open set $V \subseteq N$ then the composition $f \circ \phi$ is harmonic on $\phi^{-1}(V) \subseteq M$.

In [4, 5] P. Baird and the second author studied harmonic morphisms from a three-dimensional simply-connected space form to a surface and obtained a complete local and global classification of them. The classification of three-dimensional Euclidean and spherical space forms is well-known cf. [13, Chapters 3 and 7]. This motivates us to study the global classification of harmonic morphisms from three-dimensional *non-simply-connected* Euclidean and spherical space forms to a surface. In this paper, we obtain a description of *all* harmonic morphisms from any three-dimensional Euclidean and spherical space form to a surface, namely that any such harmonic morphism is the composition of a standard harmonic morphism (see Remark 3.2) and a weakly conformal map.

Section 2 gives some basic facts on harmonic morphisms and conformal foliations. Section 3 contains the main results of the paper. Section 4 lists the standard harmonic morphisms from all the spherical and orientable Euclidean space forms. We shall make use of standard properties of orbifolds and Seifert fibre spaces; the reader is referred to [11, 12] for these.

2. Background material on harmonic morphisms

2.1. Harmonic morphisms.

Let $(M^m, \langle \cdot, \cdot \rangle^M)$ and $(N^n, \langle \cdot, \cdot \rangle^N)$ be smooth (C^∞) Riemannian manifolds of dimensions m, n respectively.

Definition 2.1. A smooth map $\phi \colon M^m \to N^n$ is called a *harmonic morphism* if, for every real-valued function f which is harmonic on an open subset V of N with $\phi^{-1}(V)$ non-empty, $f \circ \phi$ is a real-valued harmonic function on $\phi^{-1}(V) \subset M$.

For a smooth map $\phi \colon M^m \to N^n$, let $C_\phi = \{x \in M \mid \text{rank } d\phi_x < n\}$ be its *critical set*. The points of the set $M \setminus C_\phi$ are called *regular points*. For each $x \in M \setminus C_\phi$, the *vertical space* $T_x^V M$ at x is defined by $T_x^V M = \text{Ker } d\phi_x$. The

horizontal space $T_x^H M$ at x is given by the orthogonal complement of $T_x^V M$ in $T_x M$ so that $T_x M = T_x^V M \oplus T_x^H M$.

Definition 2.2. A smooth map $\phi \colon (M^m, \langle \cdot, \cdot \rangle^M) \to (N^n, \langle \cdot, \cdot \rangle^N)$ is called *horizontally (weakly) conformal* if $d\phi = 0$ on C_ϕ and the restriction of ϕ to $M \setminus C_\phi$ is a conformal submersion, that is, for each $x \in M \setminus C_\phi$, the differential $d\phi_x \colon T_x^H M \to T_{\phi(x)} N$ is conformal and surjective. This means that there exists a function $\lambda : M \setminus C_\phi \to \mathbb{R}^+$ such that

$$\langle d\phi(X), d\phi(Y) \rangle^N = \lambda^2 \langle X, Y \rangle^M \quad \forall X, Y \in T^H M.$$

By setting $\lambda = 0$ on C_ϕ, we can extend $\lambda : M \to \mathbb{R}_0^+$ to a continuous function on M such that λ^2 is smooth, in fact $\lambda^2 = \|d\phi\|^2/n$. The function $\lambda : M \to \mathbb{R}_0^+$ is called the *dilation* of the map ϕ.

The characterization obtained by B. Fuglede [7] and T. Ishihara [9], states that *harmonic morphisms are precisely the harmonic maps which are horizontally (weakly) conformal.*

When the codomain is two-dimensional, harmonic morphisms $\phi \colon M^m \to N^2$ have special features. An important property is *conformal invariance*, explained as follows.

Proposition 2.3. *Let N_1^2, N_2^2 be surfaces, i.e. 2-dimensional Riemannian manifolds. Let $\phi \colon M^m \to N_1^2$ be a harmonic morphism and $\psi : N_1^2 \to N_2^2$ a weakly conformal map. Then $\psi \circ \phi : M^m \to N_2^2$ is a harmonic morphism.*

Thus the notion of a harmonic morphism to a surface (N^2, h) depends only on the conformal equivalence class of h. In particular, the concept of a harmonic morphism to a Riemann surface is well-defined.

The reader is referred to [7, 1, 14], for further basic properties of harmonic morphisms.

2.2. **Harmonic morphisms and conformal foliations.**

The fibres of a submersion $\phi \colon M^m \to N^n$ define a foliation on M whose leaves are the connected components of the fibres and any foliation is given locally this way. A foliation on (M^m, g) is called *conformal* (respectively *Riemannian*) if it is given locally by conformal (respectively Riemannian) submersions from open subsets of M^m. For alternative definitions and more information see [14, Sec. 3].

The relationship between harmonic morphisms and conformal foliations is the following: By [7, 9] the fibres of a submersive harmonic morphism $\phi : M^m \to N^n$ define a conformal foliation on M. In case N is a surface, a submersion $\phi : M^m \to N^2$ is a harmonic morphism if and only if it is horizontally conformal with minimal fibres [3, 8]. (In fact, this remains true for any non-constant map, see [15]). When M is a 3-dimensional manifold, the conformal foliation defined by a harmonic morphism can be extended over the critical points of ϕ and we have

Proposition 2.4. [6] *Let $\phi : M^3 \to N^2$ be a non-constant harmonic morphism. Then the fibres of ϕ form a conformal foliation $\mathcal{F}$ by geodesics of M^3.*

It follows that, *locally*, ϕ is a submersion to an open subset of $\mathbb{C}$ followed by a weakly conformal map (cf. [5]).

2.3. Harmonic morphisms from three-dimensional simply-connected space forms to a surface.

A complete classification of harmonic morphisms from a three-dimensional *simply-connected* space form was obtained by P. Baird and the second author in [4, 5]. To state their results we consider the following *standard examples of harmonic morphisms from three-dimensional simply-connected space forms*:

1. *Orthogonal projection* $\pi_1 : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $\pi_1(x_1, x_2, x_3) = (x_1, x_2)$.
2. The *Hopf map* $\pi_2 : \mathbb{S}^3 \to \mathbb{S}^2$. Identifying $\mathbb{S}^2$ with the one-point compactification $\mathbb{C} \cup \infty$ of the complex numbers via stereographic projection and taking $\mathbb{S}^3$ as $\mathbb{S}^3 = \left\{ (z_1, z_2) \in \mathbb{C}^2 : \|z_1\|^2 + \|z_2\|^2 = 1 \right\}$, then the Hopf map is given by $\pi_2(z_1, z_2) = z_1/z_2$.
3. *Orthogonal projection* $\pi_3 : \mathbb{H}^3 \to \mathbb{H}^2$ defined as follows: Consider the Poincaré disc model of $\mathbb{H}^3$ i.e. $\mathbb{H}^3$ identified with $\mathbf{D}^3 = \{ x = (x_1, x_2, x_3) : \|x\| < 1 \}$ with the metric $4 \sum dx_i^2 / (1 - \|x\|^2)^2$. Identify $\mathbb{H}^2$, via the Poincaré model, with the equatorial disc $x_3 = 0$. Then π_3 is defined by projecting each $x \in \mathbb{H}^3$ to $\mathbb{H}^2$ along the unique hyperbolic geodesic through x which meets $\mathbb{H}^2$ orthogonally.
4. *Orthogonal projection* $\pi_4 : \mathbb{H}^3 \to \mathbb{C}$ *to the plane at infinity* defined by considering the upper half-space model of $\mathbb{H}^3 = \{ (x_1, x_2, x_3) : x_3 > 0 \}$ with the metric $\sum dx_i^2 / x_3^2$ and setting $\pi_4(x_1, x_2, x_3) = x_1 + i x_2$.

By Proposition 2.4, the fibres of a non-constant harmonic morphism from a three-dimensional manifold M^3 form a conformal foliation $\mathcal{F}$ by geodesics of M^3; the conformal foliations corresponding to the harmonic morphisms defined in above examples are, respectively,

1. $\mathcal{F}_1$, the foliation of $\mathbb{R}^3$ by vertical straight lines;

2. $\mathcal{F}_2$, the foliation of $\mathbb{S}^3$ by great circles given by the intersection of 1-dimensional complex subspaces of $\mathbb{C}^2$ with $\mathbb{S}^3 \subset \mathbb{C}^2$;

3. $\mathcal{F}_3$, the foliation of $\mathbb{H}^3$ by geodesics orthogonal to the equatorial disc, in the Poincaré disc model of $\mathbb{H}^3$;

4. $\mathcal{F}_4$, the foliation of $\mathbb{H}^3$ by vertical half lines, in the upper half-space model of $\mathbb{H}^3$.

Note that $\mathcal{F}_1$, and $\mathcal{F}_2$ are, in fact, Riemannian foliations. Now we can give the classification result for harmonic morphisms from a 3-dimensional simply-connected space form $\mathbb{E}^3$ to a surface.

Theorem 2.5. [5] *Up to isometries of $\mathbb{E}^3$, a non-constant harmonic morphism of a three-dimensional simply-connected space form $\mathbb{E}^3$ to a surface N^2 is one of the examples π_i described above followed by a weakly conformal map to N^2 and the associated conformal foliation by geodesics of $\mathbb{E}^3$ is one of the examples $\mathcal{F}_i$ described above.*

2.4. The smoothing process of Baird and Wood.

The smoothing process of Baird and the second author, obtained in [6], is that an orbifold O, which is the leaf space of a Seifert fibre space without reflections can be smoothed to have a conformal structure.

Specifically, let $\left(M^3, \mathcal{F}\right)$ be a Seifert fibre space without reflections, with a C^∞ metric such that the Seifert foliation $\mathcal{F}$ is a conformal foliation by geodesics. Let L^M denote the leaf space of M^3; this is an orbifold with only cone points as singular points such that the cone points correspond to the critical fibres, cf. [11]. The horizontal conformality of ϕ at points on regular fibres gives $L^M \setminus \{\text{cone points}\}$ a conformal structure c; then away from the critical fibres, the natural projection ϕ of $\left(M^3, \mathcal{F}\right)$ onto $(L^M \setminus \{\text{cone points}\}, c)$ is a harmonic morphism. We have

Proposition 2.6. [6, Prop. 2.6] *With the above notation, there exists a unique C^∞ conformal structure on L^M, such that if we denote L^M with this conformal structure as L_s^M, the inclusion map $I : \left(L^M \setminus \{cone\ points\}, c\right) \to L_s^M$ is conformal. The composition $I \circ \phi : M^3 \to L_s^M$ is a smooth harmonic morphism.*

We need a variant of this which we can state without mention of a 3-manifold:

A *2-dimensional Riemannian orbifold* is an orbifold which is locally the quotient of a 2-dimensional Riemannian manifold by a discrete group Γ' of isometries [10, 11], it has only cone points as singularities if and only if for each p, stabilizers Γ'_p are all finite subgroups of orientation preserving isometries.

Such an orbifold has a Riemannian metric and so a conformal structure away from the cone points. Then in the same way as [6, Prop. 2.6] we have

Proposition 2.7. *Let L be a 2-dimensional Riemannian orbifold with only cone points as singularities. Then there exists smooth conformal structure on L such that, if we denote L with this conformal structure by L_s, the identity map $I : L \to L_s$ is conformal on $L \setminus \{cone\ points\}$.*

We shall call L_s the *smoothed orbifold (associated to L)*.

3. Harmonic morphisms from three-dimensional non-simply-connected Euclidean and spherical space forms to a surface

Throughout this section $\mathbb{E}_1^m$ will denote $\mathbb{R}^m$ and $\mathbb{E}_2^m$ will denote $\mathbb{S}^m$. Further Γ will denote a discrete group of isometries of $\mathbb{E}_i^3$ acting freely on $\mathbb{E}_i^3$. Note that such a Γ acts properly discontinuously [11, p 406] and the quotient $M^3 = \mathbb{E}_i^3/\Gamma$ is a Euclidean or spherical space form, i.e. a connected complete 3-dimensional Riemannian manifold of constant non-negative curvature [13, p 69]. Conversely any Euclidean or spherical 3-dimensional space form is homothetic to such a quotient.

We first describe some *standard* harmonic morphisms from such a space form to a surface:

Theorem 3.1. *Let $\pi_i : \mathbb{E}_i^3 \to \mathbb{E}_i^2$ $(i = 1, 2)$ be one of the standard harmonic morphisms in §2.3 and $\mathcal{F}_i$ be the corresponding Riemannian foliation by geodesics. Suppose that Γ is a discrete group of isometries acting freely on $\mathbb{E}_i^3$ such that*

(a): Γ *preserves* $\mathcal{F}_i$, *i.e.* Γ *maps leaves of* $\mathcal{F}_i$ *to leaves.*

Then

(i): Γ *descends through* π_i *to an action of* $\mathbb{E}_i^2$ *by a group* Γ' *of isometries,*

(ii): $\mathcal{F}$ *factors to a Riemannian foliation* $\mathcal{F}_{i,\Gamma}$ *by geodesics of the space form* $M^3 = \mathbb{E}_i^3/\Gamma$ *with leaf space* $L^M = \mathbb{E}_i^2/\Gamma'$.

Suppose further that

(b1): *for any* $p \in \mathbb{E}_i^2$ *the stabilizer* $\Gamma'_p \subset \Gamma'$ *of* p *contains no reflections,*

(b2): Γ' *acts discontinuously on* $\mathbb{E}_i^2$.

Then

(iii): $L^M = \mathbb{E}_i^2/\Gamma'$ *is a Riemannian orbifold whose only possible singularities are cone points;*

(iv): *Letting* L_s^M *denote the smoothed orbifold and* $I : L^M \to L_s^M$ *the identity map,* π_i *factors to a continuous map* $\pi_{i,\Gamma} : M^3 \to L^M$ *such that the composition* $\phi_{i,\Gamma} : M^3 \xrightarrow{\pi_{i,\Gamma}} L^M \xrightarrow{I} L_s^M$ *is a smooth harmonic morphism. We thus have a commutative diagram, where the vertical arrows are natural projections.*

Remark 3.2.

1. We shall see later that Conditions **(b1)**, **(b2)** are *necessary* for **(iii)** and **(iv)**. We shall again call the harmonic morphisms $\phi_{i,\Gamma}$ *standard harmonic morphisms.*

2. The same proposition holds with minor changes for the two standard harmonic morphisms from $\mathbb{H}^3$. However, as quotients of $\mathbb{H}^3$ have not yet been classified,

we shall not consider this case further. (Note also that there are no non-constant harmonic morphisms from compact quotients of $\mathbb{H}^3$ [2].)

Proof.

(i): Since Γ preserves $\mathcal{F}_i$ it induces an action on the leaf space $\mathbb{E}_i^2$; since Γ preserves the distance between leaves of the Riemannian foliations $\mathcal{F}_i$, this action is by a group Γ' of isometries.

(ii): This is clear since the natural projection $\mathbb{E}^3 \to \mathbb{E}^3/\Gamma$ is a Riemannian covering.

(iii): Immediate from **(i)** by definition of Riemannian orbifold.

(iv): By definition of Γ', π_i factors to a continuous map $\pi_{i,\Gamma} : M^3 \to L^M$ which is an isometry away from cone points. Since the identity map $I : L^M \to L_s^M$ is conformal away from cone points, the composition $\phi_{i,\Gamma} = I \circ \pi_{i,\Gamma} : M^3 \to L_s^M$ is C^0 and is a C^∞ harmonic morphism on $M^3 \setminus \pi_{i,\Gamma}^{-1}(\text{conepoints})$. As in [5, Theorem 2.18] it is a C^∞ harmonic morphism. Since the composition $\mathbb{E}^2 \to L^M = \mathbb{E}^2/\Gamma' \overset{I}{\to} L_s^M$ has branch points precisely at the inverse images of cone points, $\phi_{i,\Gamma}$ has critical points precisely at $\pi_{i,\Gamma}^{-1}(\text{conepoints})$.

$\square$

Remarks.

If M^3 is compact, $(M^3, \mathcal{F}_{i,\Gamma})$ is a Seifert fibre space without reflections and our construction is the same as that in [6, Prop. 2.6]. However, if M^3 is not compact, the leaves of $\mathcal{F}_{i,\Gamma}$ may not be closed and our construction is a little more general.

Example 1.

Suppose that Γ is the infinite cyclic group generated by the glide reflection $(x_1, x_2, x_3) \mapsto (-x_1, x_2, x_3 + 1)$. This preserves the standard foliation $\mathcal{F}_1$ on $\mathbb{R}^3$ and factors to the group Γ', of order 2, of isometries of $\mathbb{R}^2$ generated by the reflection $(x_1, x_2) \mapsto (-x_1, x_2)$ so that $\mathbb{R}^2/\Gamma'$ is the half plane, an orbifold with reflector line $x_2 = 0$. $M^3 = \mathbb{R}^3/\Gamma$ is the product of $\mathbb{R}$ and an infinite width Moebius strip and the foliation $\mathcal{F}_1$ factors to a Seifert fibration $\mathcal{F}_{1,\Gamma}$ on M^3 with reflections, with critical fibres the vertical lines in the plane $x_2 = 0$. However, Γ does not satisfy Condition **(b1)** and there is no harmonic morphisms from M^3 with associated foliation $\mathcal{F}_{1,\Gamma}$.

Example 2.

Suppose that Γ is the infinite cyclic group generated by the screw motion $(\mathcal{R}_\theta, t_{v_3})$ consisting of a rotation about the x_3-axis through an angle θ and a translation by one unit in the x_3-direction. Then Γ preserves the standard foliation $\mathcal{F}_1$ on $\mathbb{R}^3$ and factors to the cyclic group Γ' of isometries of $\mathbb{R}^2$ generated by rotation $\mathcal{R}_\theta$ through θ about the origin. There are two cases:

1. If $\theta/2\pi$ is irrational, Γ' is infinite cyclic, $\mathbb{R}^3/\Gamma$ is diffeomorphic to $\mathbb{R}^3$ and $\mathcal{F}_1$ factors to a foilation $\mathcal{F}_{1,\Gamma}$ of $\mathbb{R}^3/\Gamma$ by straight lines. Γ does not satisfy Condition (b2) and there is no harmonic morphism from $\mathbb{R}^3/\Gamma$ to a surface with corresponding foliation $\mathcal{F}_{1,\Gamma}$.

2. If $\theta/2\pi$ is rational, say p/q in lowest terms, Γ' is cyclic of order q generated by rotation through $2\pi/q$, $L^M = \mathbb{R}^2/\Gamma'$ is an orbifold with one cone point of angle $2\pi/q$, $\mathbb{R}^3/\Gamma$ is diffeomorphic to $\mathbb{R}^2 \times \mathbb{S}^1$ and $\mathcal{F}_1$ factors to a Seifert fibration $\mathcal{F}_{1,\Gamma}$ of $\mathbb{R}^3/\Gamma$ by circles. L_s^M can be identified with $\mathbb{C}$ via the homeomorphism $\mathbb{R}^2/\Gamma' \to \mathbb{R}^2 = \mathbb{C}$, $[z] \mapsto z^q$ and the resulting standard harmonic morphism is given by

$$\mathbb{R}^3/\Gamma \to \mathbb{R}^2, \quad [(z,t)] \mapsto z^q.$$

We next show that, up to postcomposition with weakly conformal maps, our standard harmonic morphisms give *all* harmonic morphisms from complete flat or spherical 3-dimensional space forms. Let $\mathbf{G}_i$ denote the set of all discrete groups of isometries acting freely on $\mathbb{E}_i^3$ and satisfying (a), (b1) and (b2) of Theorem 3.1.

Theorem 3.3. *Let M^3 be a complete 3-dimensional space form of non-negative curvature and let $\phi : M^3 \to N^2$ be a non-constant harmonic morphism. Then*

1. *M^3 is homothetic to $\mathbb{E}_i^3/\Gamma$ for some $\Gamma \in \mathbf{G}_i$*
2. *ϕ is the composition of a homothety $M^3 \to \mathbb{E}_i^3/\Gamma$, a standard harmonic morphism $\phi_{i,\Gamma} : \mathbb{E}_i^3/\Gamma \to L_s^M$ and a weakly conformal map $L_s^M \to N^2$.*

Proof. As in [13, p 69] M^3 is homothetic to $\mathbb{E}_i^3/\Gamma$ for some discrete group Γ of isometries acting freely on $\mathbb{E}_i^3$. Identifying M^3 with $\mathbb{E}_i^3/\Gamma$, the composition $\hat{\phi} : \mathbb{E}_i^3 \to \mathbb{E}_i^3/\Gamma \to N^2$ is a harmonic morphism. By [4] (see Theorem 2.5 above) after applying an isometry of $\mathbb{E}_i^3$, this is the composition of the standard harmonic morphism $\pi_i : \mathbb{E}_i^3 \to \mathbb{E}_i^2$ with a weakly conformal map $\hat{\xi} : \mathbb{E}_i^2 \to N^2$. Γ preserves

the corresponding foliation $\mathcal{F}_i$ which therefore descends to a Riemannian foliation $\mathcal{F}_{i,\Gamma}$ by geodesics of $\mathbb{E}_i^3/\Gamma \simeq M^3$. Thus Γ satisfies Condition **(a)**.

Further Γ' acts by isometries on $\mathbb{E}_i^2$ and, by diagram, $\hat{\xi}$ factors to a continuous map $\xi : \mathbb{E}_i^2/\Gamma' \to N^2$ conformal away from critical points so that we have a commutative diagram.

$$
\begin{array}{ccc}
\mathbb{E}_i^3 & \xrightarrow{\hat{\pi}_i} & \mathbb{E}_i^2 \\
\downarrow{\pi} & & \downarrow{\pi'} \\
M^3 = \mathbb{E}_i^3/\Gamma & \xrightarrow{\pi_{i,\Gamma}} & L^M = \mathbb{E}_i^2/\Gamma'
\end{array}
$$

with maps $\hat{\phi}$, $\hat{\xi}$ to N^2 and ϕ, ξ to N^2.

Now, if Γ' did not act discontinuously, then, in the neighbourhood of some $p \in M^3$ there would be an infinite sequence of line segments mapped by ϕ to the same point, which is impossible by the local form of a harmonic morphism (see §2.4), thus Condition **(b2)** is satisfied..

Further for any $p \in \mathbb{E}_i^2$ the action of its stabilizer Γ'_p preserves the fibres of $\hat{\xi}$, since any point is either a regular point of $\hat{\xi}$ or a branch point of order $k \geq 2$, this means Γ'_p lies in a cyclic subgroup of order $k \geq 1$ of $SO(2)$ so that Condition **(b1)** is satisfied. We have shown that $\Gamma \in \mathbf{G}_1$.

Then $\mathbb{E}_i^2/\Gamma'$ is an orbifold with only cone points. Note that it is the leaf space L^M of $\mathcal{F}_{i,\Gamma}$, it can thus be smoothed as above and the resulting harmonic morphism $\phi_{i,\Gamma} : \mathbb{E}_i^3/\Gamma \to L^M \xrightarrow{I} L_s^M$ is a standard harmonic morphism. Clearly $\xi : L_s^M \to N^2$ is C^0, and C^∞ and conformal away from the cone points, therefore conformal everywhere and the Theorem is proved. $\qquad\square$

To complete our description of harmonic morphisms from Euclidean and spherical space forms M^3 we list the standard harmonic morphisms for these two cases.

4.1. **The Euclidean case.** Recall that $\mathbf{G}_1$ denotes the set of all discrete groups of isometries acting freely on $\mathbb{R}^3$ and satisfying **(a)**, **(b1)** and **(b2)** of Theorem 3.1.

Theorem 4.1. *Let M^3 be a complete flat space form.*

1. *Any harmonic morphism from M^3 to a surface N^2 is the composition of an isometry $M^3 \to \mathbb{R}^3/\Gamma$, a standard harmonic morphism $\phi_{1,\Gamma}$ for some $\Gamma \in \mathbf{G}_1$ and a weakly conformal map $L_s^M \to N^2$.*

2. *The standard harmonic morphisms are given by*

$$\phi_{1,\Gamma} = I \circ \pi_{1,\Gamma} : \mathbb{R}^3/\Gamma \longrightarrow \mathbb{R}^2/\Gamma' = L^M \xrightarrow{I} L_s^M$$

   where $\Gamma \in \mathbf{G}_1$.

3. *For each isometry type of M^3, there are (non-constant) standard harmonic morphisms. We list these for orientable M^3. In the following list, with notations explained below, for each of the 10 diffeomorphism types A of M^3, we list the possible groups Γ and the parameters they depend on, then the different choices (a), (b), ... of parameters which ensure that $\Gamma \in \mathbf{G}_1$. For each such choice, we list the corresponding Γ' and L^M.*

M^3 *non-compact:*

1. $A = \mathcal{E}$, $M^3 \approx \mathbb{R}^3$, $\Gamma = \{e\}$:
   (a) $\Gamma' = \{e\}$, $L^M = \mathbb{R}^2$.

2. $A = \mathcal{J}_1^\theta$, $M^3 \approx \mathbb{S}^1 \times \mathbb{R}^2$, $\Gamma = \langle (R_\theta(v), t_v) \rangle$ where $\theta \in \mathbb{R}$ and $v \in \mathbb{R}^3$ is non-zero:
   (a) $v \parallel e_3$, $\theta = 0$: $\Gamma' = \{e\}$, $L^M = \mathbb{R}^2$.
   (b) $v \parallel e_3$, $\theta = \pi$: $\Gamma' = \langle R_\pi \rangle$, $L^M = \mathbb{R}^2(2) = $ *cone of angle* π.
   (c) $v \parallel e_3$, $\theta \neq 0, \pi$ and $\theta = 2\pi p/q$ where $p, q \in \mathbb{Z}$, $(p, q) = 1$, $q \neq 0$: $\Gamma' = \langle R_\theta \rangle$, $L^M = \mathbb{R}^2(q) = $ *cone of angle* $2\pi/q$.
   (d) $v \nparallel e_3$, $\theta = 0$: $\Gamma' = \langle t_{\pi(v)} \rangle$, $L^M = $ *cylinder*.
   (e) $v \in e_3^\perp$, $\theta = \pi$: $\Gamma' = \langle (S_v, t_v) \rangle$, $L^M = $ *Möbius band*.

3. $A = \mathcal{T}_1$, $M^3 \approx \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R}$, $\Gamma = \langle t_{v_1}, t_{v_2} \rangle$ where $v_1, v_2 \in \mathbb{R}^3$ are linearly independent:

   (a) $e_3 \notin \mathrm{span}\{v_1, v_2\}$: $\Gamma' = \langle t_{\pi(v_1)}, t_{\pi(v_2)} \rangle$, $L^M = $ 2-torus.

   (b) $e_3 \in \mathrm{span}\{v_1, v_2\}$ and $\{\pi(v_1), \pi(v_2)\}$ rationally related: $\Gamma' = \langle t_w \rangle$ where $\mathrm{span}_\mathbb{Z}\{w\} = \mathrm{span}_\mathbb{Z}\{\pi(v_1), \pi(v_2)\} \in \mathbb{R}^2$, $L^M = $ cylinder.

4. $A = \mathcal{K}_1$, $\Gamma = \langle t_{v_1}, (R_\pi(v_2), t_{v_2}) \rangle$ where $v_1, v_2 \in \mathbb{R}^3$ are linearly independent.

   (a) $v_1 \not\parallel e_3$, $v_2 \in e_3^\perp$: $\Gamma' := \langle t_{\pi(v_1)}, (S_{v_2}, t_{v_2}) \rangle$, $L^M = $ Klein bottle.

   (b) $v_1 \parallel e_3$, $v_2 \in e_3^\perp$, $\Gamma' = \langle (S_{v_2}, t_{v_2}) \rangle$, $L^M = $ Möbius band.

   (c) $v_2 \parallel e_3$: $\Gamma' = \langle t_{\pi(v_1)}, R_\pi \rangle$, $L^M = D^2(2, 2)$.

M^3 compact:

5. $A = \mathcal{G}_1$, $M^3 = $ 3-torus, $\Gamma = \langle t_{v_1}, t_{v_2}, t_{v_3} \rangle$ where $v_1, v_2, v_3 \in \mathbb{R}^3$ are linearly independent:

   (a) $\{\pi(v_1), \pi(v_2), \pi(v_3)\}$ rationally related: $\Gamma' = \langle t_{w_1}, t_{w_2} \rangle$ where $\mathrm{span}_\mathbb{Z}\{w_1, w_2\} = \mathrm{span}_\mathbb{Z}\{\pi(v_1), \pi(v_2), \pi(v_3)\}$, $L^M = $ 2-torus.

6. $A = \mathcal{G}_2$, $\Gamma = \langle (R_\pi(v_1), t_{v_1/2}), t_{v_1}, t_{v_2}, t_{v_3} \rangle$ where $v_1, v_2, v_3 \in \mathbb{R}^3$ are linearly independent and $v_1 \perp \mathrm{span}\{v_2, v_3\}$:

   (a) $v_1 \in e_3^\perp$, $\{\pi(v_2), \pi(v_3)\}$ rationally related: $\Gamma' = \langle (S_{v_1}, v_1), t_w \rangle$ where $\mathrm{span}_\mathbb{Z}\{w\} = \mathrm{span}_\mathbb{Z}\{\pi(v_2), \pi(v_3)\}$, $L^M = $ Klein bottle.

   (b) $v_1 \parallel e_3$: $\Gamma' = \langle R_\pi, t_{v_2}, t_{v_3} \rangle$, $L^M = \mathbb{S}^2(2, 2, 2, 2)$.

7. $A = \mathcal{G}_3$, $\Gamma = \langle (R_{2\pi/3}(v_1), t_{v_1/3}), t_{v_2}, t_{v_3}, t_{v_3} \rangle$ where $v_1, v_2, v_3 \in \mathbb{R}^3$ are linearly independent, $v_1 \perp \mathrm{span}\{v_2, v_3\}$, $\|v_2\| = \|v_3\|$, $\mathrm{angle}(v_2, v_3) = 2\pi/3$:

   (a) $v_1 \parallel e_3$: $\Gamma' = \langle R_{2\pi/3}, t_{v_2}, t_{v_3} \rangle$, $L^M = \mathbb{S}^2(3, 3, 3)$.

8. $A = \mathcal{G}_4$, $\Gamma = \langle (R_{\pi/2}(v_1), t_{v_1/4}), t_{v_1}, t_{v_2}, t_{v_3} \rangle$ where $v_1, v_2, v_3 \in \mathbb{R}^3$ are mutually orthogonal with $\|v_2\| = \|v_3\|$:

   (a) $v_1 \parallel e_3$: $\Gamma' = \langle R_{\pi/2}, t_{v_2}, t_{v_3} \rangle$, $L^M = \mathbb{S}^2(2, 4, 4)$.

9. $A = \mathcal{G}_5$, $\Gamma = \langle (R_{\pi/6}(v_1), t_{v_1/6}), t_{v_1}, t_{v_2}, t_{v_3} \rangle$ where $v_1, v_2, v_3 \in \mathbb{R}^3$ are linearly independent, $v_1 \perp \mathrm{span}\{v_2, v_3\}$, $\|v_2\| = \|v_3\|$ and $\mathrm{angle}(v_2, v_3) = \pi/6$:

   (a) $v_1 \parallel e_3$: $\Gamma' = \langle R_{\pi/6}, t_{v_2}, t_{v_3} \rangle$, $L^M = \mathbb{S}^2(2, 3, 6)$.

10. $A = \mathcal{G}_6$, $\Gamma = \langle (R_\pi(v_1), t_{v_1/2}), (R_\pi(v_2), t_{(v_2+v_3)/2}), (R_\pi(v_3), t_{(v_1+v_2+v_3)/2}), t_{v_1}, t_{v_2}, t_{v_3} \rangle$ where v_1, v_2, v_3 are mutually orthogonal:

   (a) $v_1 \parallel e_3$: $\Gamma' = \langle R_\pi, (S_{v_2}, t_{(v_2+v_3)/2}), (S_{v_3}, t_{(v_2+v_3)/2}), t_{v_2}, t_{v_3} \rangle$, $L^M = \mathbb{P}^2(2, 2)$.

(b) $v_2 \parallel e_3$: $\Gamma' = \left\langle (S_{v_1}, t_{v_1/2}), (R_\pi, t_{v_3/2}), (S_{v_3}, t_{(v_1+v_3)/2}), t_{v_1}, t_{v_3} \right\rangle$, $L^M = \mathbb{P}^2(2, 2)$.

(c) $v_3 \parallel e_3$: $\Gamma' = \left\langle (S_{v_1}, t_{v_1/2}), (S_{v_2}, t_{v_2/2}), (R_\pi, t_{(v_1+v_2)/2}), t_{v_1}, t_{v_2} \right\rangle$, $L^M = \mathbb{P}^2(2, 2)$.

Notation

For $v \in \mathbb{R}^3$, t_v denotes translation in $\mathbb{R}^3$ through v, and for $v \neq 0$, $\theta \in \mathbb{R}$, $R_\theta(v)$ denotes rotation in $\mathbb{R}^3$ through an angle θ about v.

For $v \in \mathbb{R}^2$, t_v denotes translation in $\mathbb{R}^2 = e_3^\perp$ through v, for $v \neq 0$, S_v denotes reflection in v and, for $\theta \in \mathbb{R}$, R_θ denotes rotation about the origin through an angle θ.

Bracketted pairs (A, t) denote A followed by t, thus, for example, $(R_\theta(v), t_v)$ is the screw motion through θ along v.

We say that vectors v_i, $(i = 1, \cdots, k)$ are *rationally related* if they are linearly dependent over $\mathbb{Q}$. If $k = 2$ and the v_i are not all parallel, this means that the corresponding translations t_{v_i} generate a 2-dimensional lattice.

$N(r, s, \cdots)$ denotes the orbifold with underlying surface N and cone points of orders $r, s, \cdots$.

Proof.

Parts 1. and 2. follow from Theorems 3.1 and 3.3 noting that we can scale the homothety of M^3 to $\mathbb{R}^3$ to make it an isometry. For Part 3, the discrete groups Γ acting freely on $\mathbb{R}^3$ and giving orientable quotients $\mathbb{R}^3/\Gamma$ are classified in [13, Theorem 3.5.1, 3.5.5] up to affine diffeomorphism, we add parameters according to the remarks at the end of [13, §3.5] to obtain all groups Γ up to translational equivalence. Then we determine the values of those parameters for which $\Gamma \in \mathbf{G}_1$, using the following simple facts: (i) condition (b1) is automatic for $\mathbb{R}^3/\Gamma$ orientable, (ii) a rotation through an angle $\theta \in (0, 2\pi)$ about a vector v preserves the e_3-direction if and only if either $v \parallel e_3$ or $v \perp e_3$ and $\theta = \pi$. The induced group Γ' and the resulting orbifold $L^M = \mathbb{R}^2/\Gamma'$ are then calculated. $\square$

Remarks.

1. Each of the 10 diffeomorphism types represents exactly one *affine* diffeomorphism type of M^3 except for $\mathcal{J}_1^\theta$, where the affine diffeomorphism type is parameterized by $\theta \mod \pi$.

2. For 1, 2(d), 2(e), 3(a), 4(a) the foliation $\mathcal{F}_{1,\Gamma}$ defined by Γ has fibres which are lines, for the other cases they are circles and $(M^3, \mathcal{F}_{1,\Gamma})$ is a Seifert fibre space.

3. A similar list can be given for M^3 non-orientable.

4.2. The spherical case.

First note that, since orientation reversing transformations have fixed points, any discrete subgroup of $\mathrm{Isom}(\mathbb{S}^3) = O(4)$ acting freely is a finite subgroup of $SO(4)$.

Identify $\mathbb{R}^4$ with the quaternions and $\mathbb{S}^3$ with the unit quaternions. Let $\psi : \mathbb{S}^3 \to SO(3)$ be the standard double cover given by $\psi(q) = a \mapsto qaq^{-1}$ and let $\phi : \mathbb{S}^3 \times \mathbb{S}^3 \to SO(4)$ be the double cover given by $\phi(q_1, q_2) = x \mapsto q_1 x q_2^{-1}$. Let $\Gamma_1 = \phi(\mathbb{S}^1 \times \mathbb{S}^3)$ and $\Gamma_2 = \phi(\mathbb{S}^3 \times \mathbb{S}^1)$. Let $p : SO(4) \to SO(3) \times SO(3)$ be the unique map defined by $\psi \times \psi = p \circ \phi$ so that p is a surjective homomorphism with $\mathrm{Ker}\, p = \{I, -I\} \subset SO(4)$, and let $H_i \subset SO(3)$ $(i = 1, 2)$ be the projections of $H = p(\Gamma)$ onto the two factors. Then a finite subgroup Γ of $SO(4)$ preserves the standard Hopf foliation $\mathcal{F}_2$ if and only if either $\Gamma \subset \Gamma_1$, or $\Gamma \subset \Gamma_2$ with H_1 dihedral with cyclic subgroup in $\psi(\mathbb{S}^1)$ and H_2 is cyclic [11]. In the first case H_1 is cyclic and H_2 is cyclic, dihedral or the symmetry group of a regular solid namely, the tetrahedral, octahedral or icosahedral groups **T**, **O** or **I**. Further the induced action of $(q_1, q_2) \in \mathbb{S}^1 \times \mathbb{S}^3$ on the leaf space $\mathbb{S}^2$ is given by $\psi(q_2)$ so that $L^M = \mathbb{S}^2/H_2$. In the second case, we must additionally factor out by the action of an element of order 2 in H_1 giving an orbifold with underlying space real projective 2-space $\mathbb{P}^2$. Thus we have

Theorem 4.2. *Let M^3 be a spherical space form.*

1. *Any harmonic morphism from a complete spherical space form M^3 to a surface N^2 is the composition of a homothety $M^3 \to \mathbb{S}^3/\Gamma$, a standard harmonic morphism $\phi_{2,\Gamma}$ for some finite subgroup Γ of Γ_1 and a weakly conformal map $L_s^M \to N^2$.*

2. *The standard harmonic morphisms are given by*

$$\phi_{2,\Gamma} = I \circ \pi_{2,\Gamma} : \mathbb{S}^3/\Gamma \longrightarrow \mathbb{S}^2/\Gamma' = L^M \overset{I}{\longrightarrow} L_s^M$$

where Γ is a finite subgroup of Γ_1.

3. *For each isometry type of M^3 there are (non-constant) standard harmonic morphisms. In the following table, we categorize these according to the subgroups H_1 and H_2; we list the corresponding quotient $\mathbb{S}^3/\Gamma$ and orbifold L^M.*

H_1	H_2	$M^3 = \mathbb{S}^3/\Gamma$	L^M
$\mathbb{Z}_p$	$\mathbb{Z}_q$	*Lens spaces*	$\mathbb{S}^2(q,q)$
$\mathbb{Z}_p$	$\mathbf{D_m}$	*Prism spaces*	$\mathbb{S}^2(2,2,m)$
$\mathbf{D}_m$	$\mathbb{Z}_q$	*Prism spaces*	$\mathbb{P}^2(q)$
$\mathbb{Z}_p$	$\mathbf{T}$	*Tetrahedral spaces*	$\mathbb{S}^2(2,3,3)$
$\mathbb{Z}_p$	$\mathbf{O}$	*Octahedral spaces*	$\mathbb{S}^2(2,3,4)$
$\mathbb{Z}_p$	$\mathbf{I}$	*Icosahedral spaces*	$\mathbb{S}^2(2,3,5)$

Proof. Parts 1. and 2. follow from Theorems 3.1 and 3.3.

For Part 3., that there are standard harmonic morphisms for any isometry type follows from the fact that any finite subgroup of $SO(4)$ acting freely on $\mathbb{S}^3$ is conjugate in $O(4)$ to a subgroup of Γ_1 [11]. □

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