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Symmetry analysis of wave equation on sphere

H. Azad, M.T. Mustafa *

Department of Mathematical Sciences, King Fahd University of Petroleum & Minerals, Dhahran 31261, Saudi Arabia

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Abstract

The symmetry classification problem for wave equation on sphere is considered. Symmetry algebra is found and a classification of its subalgebras, up to conjugacy, is obtained. Similarity reductions are performed for each class, and some examples of exact invariant solutions are given.

Keywords: Group invariant solutions; Lie symmetries; Partial differential equations; Symmetry analysis

1. Introduction

One of the significant applications of Lie symmetry groups is to achieve a complete classification of symmetry reductions of partial differential equations. The symmetry properties and reductions of most of the fundamental equations of mathematical physics, with flat background metric, have been well investigated, cf. [10–12]. In particular, the symmetry classification problem for a number of wave equations in flat space has been extensively studied [1–3,5,8,12,14].

This work is part of a research program to investigate natural equations of physics with non-flat background metric, for example, of non-zero constant curvature. Here, we give a complete symmetry analysis of wave equation on sphere.

The aim is to classify symmetry reductions of the wave equation on sphere, which is

$$u_{tt} = u_{xx} + (\cot x)u_x + \frac{1}{\sin^2 x}u_{yy}. \tag{1.1}$$

* Corresponding author.
E-mail addresses: hassanaz@kfupm.edu.sa (H. Azad), tmustafa@kfupm.edu.sa (M.T. Mustafa).

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As described in [15], the classification of group invariant solutions requires a classification of subalgebras of the symmetry algebra into conjugacy classes under the adjoint action of the symmetry group. The symmetry algebra of Eq. (1.1) is determined in Section 2. The classification of subalgebras is also carried out in Section 2, which is then utilized in Section 3 to classify the symmetry reductions of Eq. (1.1). For the cases where Eq. (1.1) is reduced to ODE, the symmetry solutions of Eq. (1.1) are also discussed. As the symmetry methods are local, it is understood that all solutions are considered only locally.

2. The symmetry algebra and classification of subalgebras

The method of determining the classical symmetries of a partial differential equation is standard which is described in many books, e.g., [6,9,13,16]. To obtain the symmetry algebra of PDE (1.1), we take the infinitesimal generator of symmetry algebra of the form

$$X = \xi_1(x,y,t,u)\frac{\partial}{\partial x} + \xi_2(x,y,t,u)\frac{\partial}{\partial y} + \xi_3(x,y,t,u)\frac{\partial}{\partial t} + \varphi_1(x,y,t,u)\frac{\partial}{\partial u}.$$

Using the invariance condition, i.e., applying the 2nd prolongation $X^{[2]}$ to Eq. (1.1), yields the following system of 13 determining equations. The computations were performed using the package *MathLie* [4].

$$\begin{aligned}
&(\xi_1)_u = 0,\\
&(\xi_2)_u = 0,\\
&(\xi_3)_u = 0,\\
&(\phi_1)_{u,u} = 0,\\
&\cot x(\xi_3)_x - (\xi_3)_{t,t} + (\xi_3)_{x,x} + \csc^2 x(\xi_3)_{y,y} + 2(\phi_1)_{t,u} = 0,\\
&\cot x(\xi_2)_x - (\xi_2)_{t,t} + (\xi_2)_{x,x} + \csc^2 x(\xi_2)_{y,y} - 2\csc^2 x(\phi_1)_{y,u} = 0,\\
&\csc^2 x\xi_1 + \cot x(\xi_1)_x - 2\cot x(\xi_3)_t - (\xi_1)_{t,t} + (\xi_1)_{x,x} + \csc^2 x(\xi_1)_{y,y} - 2(\phi_1)_{x,u} = 0,\\
&-\cot x(\phi_1)_x + (\phi_1)_{t,t} - (\phi_1)_{x,x} - \csc^2 x(\phi_1)_{y,y} = 0,\\
&(\xi_1)_t - (\xi_3)_x = 0,\\
&\csc^2 x(\xi_1)_y + (\xi_2)_x = 0,\\
&-(\xi_1)_x + (\xi_3)_t = 0,\\
&(\xi_2)_t - \csc^2 x(\xi_3)_y = 0,\\
&\cot x\xi_1 + (\xi_2)_y - (\xi_3)_t = 0.
\end{aligned}$$

The solution of determining equations gives the following infinitesimals,

$$\begin{aligned}
&\xi_1 = k_3\cos y + k_4\sin y,\\
&\xi_2 = \cot x\{k_4\cos y - k_3\sin y\} + k_5,\\
&\xi_3 = k_1,\\
&\phi_1 = k_2 u + f(x,y,t)
\end{aligned}$$

where $f(x,y,t)$ is a function satisfying Eq. (1.1).

Hence the associated symmetry algebra of Eq. (1.1) is spanned by the vector fields

$$X_1 = \frac{\partial}{\partial t},$$
$$X_2 = u\frac{\partial}{\partial u},$$
$$X_3 = \cos y\frac{\partial}{\partial x} - \cot x \sin y\frac{\partial}{\partial y},$$
$$X_4 = \sin y\frac{\partial}{\partial x} + \cot x \cos y\frac{\partial}{\partial y},$$
$$X_5 = \frac{\partial}{\partial y},$$
$$X_f = f\frac{\partial}{\partial u}.$$

The commutation relations of the Lie algebra $\mathcal{G}$, determined by X_1, X_2, X_3, X_4, X_5, are shown in Table 1. The radical of $\mathcal{G}$ is $R = \langle X_1, X_2\rangle$ and Levi decomposition of $\mathcal{G}$ is given by

$$\mathcal{G} = \langle X_1, X_2\rangle \oplus \langle X_3, X_4, X_5\rangle$$

where the Lie algebra $\langle X_3, X_4, X_5\rangle$ is $so(3)$.

We use the scheme of Ovsiannikov [15, Section 14] for the classification of subalgebras of $\mathcal{G}$ (up to conjugacy).

There is one class of 1-dimensional subalgebras of $so(3) = \langle X_3, X_4, X_4\rangle$, whose representative is taken as X_5. This is because $SO(3)$ operates as isometries for the (negative definite) Killing form, and the image of $SO(3)$ is therefore the connected component of the conjugacy group. For the classification up to conjugacy of 2-dimensional subalgebras of $\mathcal{G} = so(3) \oplus R$, where R is 2-dimensional radical, we note that if $\mathfrak{L}$ is 2-dimensional subalgebra then its image in

Table 1
Commutator table for the Lie algebra $\mathcal{G}$

	X_1	X_2	X_3	X_4	X_5
X_1	0	0	0	0	0
X_2	0	0	0	0	0
X_3	0	0	0	$-X_5$	X_4
X_4	0	0	X_5	0	$-X_3$
X_5	0	0	$-X_4$	X_3	0

Table 2
Classification of subalgebras of symmetry algebra $\mathcal{G}$

Dimension	Subalgebra
1-dimensional subalgebra	$\mathfrak{L}_1 = \langle aX_1 + bX_2 + X_5\rangle$ $\mathfrak{L}_2 = \langle aX_1 + bX_2\rangle$
2-dimensional subalgebra	$\mathfrak{L}_3 = \langle aX_1 + bX_2, X_5\rangle$ $\mathfrak{L}_4 = \langle X_1, X_2\rangle$
3-dimensional subalgebra	$\mathfrak{L}_5 = \langle X_3, X_4, X_5\rangle$ $\mathfrak{L}_6 = \langle X_1, X_2, X_5\rangle$
4-dimensional subalgebra	$\mathfrak{L}_7 = \langle aX_1 + bX_2, X_3, X_4, X_5\rangle$

$\mathcal{G}/R \cong so(3)$ is either zero-dimensional or one-dimensional as $so(3)$ has no 2-dimensional subalgebras. The zero-dimensional case means that $\mathfrak{L} = R$. In the case that $(\mathfrak{L} + R)/R \equiv \mathfrak{L}/(\mathfrak{L} \cap R)$ is one-dimensional we see that $(\mathfrak{L} \cap R)$ is one-dimensional. For X in R we look for all elements Y of $\mathcal{G}$ so that $\langle X, Y\rangle$ form a 2-dimensional algebra. The possibilities for the commutation relations are $[X, Y] = 0$ or $[X, Y] = X$. Hence, we obtain the following representatives of the conjugacy classes of subalgebras of the symmetry algebra $\mathcal{G}$ (see Table 2).

3. Symmetry reductions for wave equation

In this section we give a classification of symmetry reductions of PDE (1.1) with respect to classification of subalgebras of $\mathcal{G}$ into conjugacy classes. It is clear that for ODEs a solvable group leads to reduction of order which is equal to dimension of the group, but for PDEs one has to introduce new dependent and independent variables so that the problem does not degenerate to constant solutions. The standard method used for this is by introduction of similarity variables. These are new independent variables as basic invariant functions which do not involve the dependent variables, and the dependent variables are defined implicitly involving invariants which contain the original variables, see [7] for details. Sections 3.1, 3.2 and 3.3 provide illustrative examples of reductions using similarity variables.

3.1. Reductions by 1-dimensional subalgebras

For each 1-dimensional subalgebra in the classification of subalgebras of $\mathcal{G}$, we obtain the similarity variables. The similarity variables are used to obtain the reduced PDE and the form of the solutions of Eq. (1.1).

3.1.1. Subalgebra $\mathfrak{L}_1 = \langle aX_1 + bX_2 + X_5\rangle$

I. $a \neq 0, b \neq 0$:

The similarity variables are

$$\xi_1 = x, \qquad \xi_2 = y - \frac{t}{a}$$

and

$$V(\xi_1, \xi_2) = \ln u - \frac{b}{a}t.$$

Substitution of similarity variables in Eq. (1.1) and using chain rule implies that the solution of Eq. (1.1) is of the form

$$u = e^{\frac{bt}{a}} e^{V(\xi_1,\xi_2)}$$

where $V(\xi_1, \xi_2)$ satisfies the following reduced PDE in 2 independent variables

$$\frac{1}{a^2}\left\{\frac{\partial^2 V}{\partial \xi_2^2} + \left(\frac{\partial V}{\partial \xi_2} - b\right)^2\right\}$$
$$= \frac{\partial^2 V}{\partial \xi_1^2} + \left(\frac{\partial V}{\partial \xi_1^2}\right)^2 + \cot\xi_1 \frac{\partial V}{\partial \xi_1} + \frac{1}{\sin^2\xi_1}\left\{\frac{\partial^2 V}{\partial \xi_2^2} + \left(\frac{\partial V}{\partial \xi_2}\right)^2\right\}.$$

II. $a \neq 0$, $b = 0$:
The similarity variables are

$$\xi_1 = x, \qquad \xi_2 = y - \frac{t}{a}$$

and

$$V(\xi_1, \xi_2) = u.$$

Substitution of similarity variables implies that the solution of Eq. (1.1) is of the form

$$u = V(\xi_1, \xi_2)$$

where $V(\xi_1, \xi_2)$ satisfies the following reduced PDE in 2 independent variables

$$\frac{1}{a^2}\frac{\partial^2 V}{\partial \xi_2^2} = \frac{\partial^2 V}{\partial \xi_1^2} + \cot\xi_1 \frac{\partial V}{\partial \xi_1} + \frac{1}{\sin^2 \xi_1}\frac{\partial^2 V}{\partial \xi_2^2}.$$

III. $a = 0$, $b \neq 0$:
The similarity variables are

$$\xi_1 = x, \qquad \xi_2 = t$$

and

$$V(\xi_1, \xi_2) = \ln u - by.$$

Hence, the solution of Eq. (1.1) is of the form

$$u = e^{by} e^{V(\xi_1, \xi_2)}$$

where $V(\xi_1, \xi_2)$ satisfies the following reduced PDE

$$\frac{\partial^2 V}{\partial \xi_2^2} + \left(\frac{\partial V}{\partial \xi_2}\right)^2 = \frac{\partial^2 V}{\partial \xi_1^2} + \left(\frac{\partial V}{\partial \xi_1}\right)^2 + \cot\xi_1 \frac{\partial V}{\partial \xi_1} + \frac{b^2}{\sin^2 \xi_1}.$$

IV. $a = 0$, $b = 0$:
In this case, the symmetry is $X = \frac{\partial}{\partial y}$, which obviously leads to y-invariant solutions. Precisely, the similarity variables are

$$\xi_1 = x, \qquad \xi_2 = t, \qquad V(\xi_1, \xi_1) = u$$

and Eq. (1.1) is reduced to PDE

$$\frac{\partial^2 V}{\partial \xi_2^2} = \frac{\partial^2 V}{\partial \xi_1^2} + \cot\xi_1 \frac{\partial V}{\partial \xi_1}.$$

3.1.2. Subalgebra $\mathfrak{L}_2 = \langle aX_1 + bX_2 \rangle$

I. $a \neq 0$, $b \neq 0$:
The similarity variables are

$$\xi_1 = x, \qquad \xi_2 = y$$

and

$$V(\xi_1, \xi_2) = \ln u - \frac{b}{a}t.$$

The solution is of the form

$$u = e^{\frac{bt}{a}} e^{V(\xi_1,\xi_2)}$$

where $V(\xi_1, \xi_2)$ satisfies the following reduced PDE

$$\frac{b^2}{a^2} = \frac{\partial^2 V}{\partial \xi_1^2} + \left(\frac{\partial V}{\partial \xi_1}\right)^2 + \cot \xi_1 \frac{\partial V}{\partial \xi_1} + \frac{1}{\sin^2 \xi_1}\left(\frac{\partial^2 V}{\partial \xi_2^2} + \left(\frac{\partial V}{\partial \xi_2}\right)^2\right).$$

II. $a \neq 0, b = 0$:
The symmetry $X = \frac{\partial}{\partial t}$ leads to time-invariant solutions. Precisely, the similarity variables are

$$\xi_1 = x, \qquad \xi_2 = y, \qquad V(\xi_1, \xi_2) = u$$

and Eq. (1.1) is reduced to

$$\frac{\partial^2 V}{\partial \xi_1^2} + \cot \xi_1 \frac{\partial V}{\partial \xi_1} + \frac{1}{\sin^2 \xi_1} \frac{\partial^2 V}{\partial \xi_2^2} = 0.$$

III. $a = 0, b \neq 0$:
In this case, the symmetry is $X = bu\frac{\partial}{\partial u}$ which leads to constant solution.

3.2. *Reduction by 2-dimensional subalgebra*

Double reduction of variables is performed for each 2-dimensional subalgebras in the classification of subalgebras of $\mathcal{G}$. This allows us to write the PDE (1.1) as ODE. The exact solutions of ODE are discussed in each case.

3.2.1. *Subalgebra $\mathfrak{L}_3 = \langle aX_1 + bX_2, X_5\rangle$*

I. $a \neq 0, b = 0$:
The similarity variables for X_5 (taken as first symmetry) are

$$\xi_1(x, y, t) = x, \quad \xi_2(x, y, t) = t \quad \text{and} \quad V(\xi_1, \xi_2) = u.$$

These reduce Eq. (1.1) to

$$\frac{\partial^2 V}{\partial \xi_2^2} = \frac{\partial^2 V}{\partial \xi_1^2} + \cot \xi_1 \frac{\partial V}{\partial \xi_1}. \tag{3.1}$$

Since the symmetry aX_1 of Eq. (1.1) commutes with X_5, it is inherited by Eq. (3.1). Hence

$$V = aX_1(\xi_1)\frac{\partial}{\partial \xi_1} + aX_1(\xi_2)\frac{\partial}{\partial \xi_2} + aX_1(V)\frac{\partial}{\partial V} = a\frac{\partial}{\partial \xi_2}$$

is a symmetry of PDE (3.1). Its similarity variables are

$$r(\xi_1, \xi_2) = \xi_1,$$
$$w(r) = V$$

which reduce PDE (3.1) to ODE

$$\frac{d^2 w}{dr^2} + \cot r \frac{dw}{dr} = 0.$$

This can be integrated to give the solution

$$w(r) = k_1 \ln(\csc r - \cot r) + k_2$$

or

$$u = k_1 \ln(\csc x - \cot x) + k_2$$

where k_1, k_2 are integration constants.

II. $a = 0, b \neq 0$:
In this case, the symmetry $bX_2 = bu\frac{\partial}{\partial u}$ leads to constant solution.

III. $a \neq 0, b \neq 0$:
The first symmetry X_5 reduces Eq. (1.1) to

$$\frac{\partial^2 V}{\partial \xi_2^2} = \frac{\partial^2 V}{\partial \xi_1^2} + \cot \xi_1 \frac{\partial V}{\partial \xi_1} \tag{3.2}$$

where

$$\xi_1(x, y, t) = x, \qquad \xi_2(x, y, t) = t, \qquad V(\xi_1, \xi_2) = u$$

are similarity variables for X_5.
The second symmetry $aX_1 + bX_2$ is inherited by PDE (3.2), as it commutes with X_5. Hence

$$V = a\frac{\partial}{\partial \xi_2} + bV\frac{\partial}{\partial V}$$

is a symmetry of PDE (3.2). Its similarity variables are

$$r(\xi_1, \xi_2) = \xi_1,$$
$$w(r) = \ln V - \frac{b}{a}\xi_2$$

which transform PDE (3.2) to ODE

$$\frac{d^2 w}{dr^2} + \left(\frac{dw}{dr}\right)^2 + \cot r \frac{dw}{dr} = \frac{b^2}{a^2}.$$

Setting $z = w'$, we obtain the first order ODE

$$z' + z^2 + \cot r z = \frac{b^2}{a^2}$$

which is Riccati equation.

3.2.2. Subalgebra $\mathfrak{L}_4 = \langle X_1, X_2 \rangle$

The symmetry $X_2 = u\frac{\partial}{\partial u}$ leads to trivial invariant solutions.

3.3. Reduction by 3- and 4-dimensional subalgebras

The step-by-step multiple reduction using similarity variables is not possible if the reduced equation fails to inherit symmetries of the original equation. This happens when step-by-step reduction is tried for $\mathfrak{L}_5$ and $\mathfrak{L}_7$. An efficient approach for such situations is to find the joint invariants of the subalgebra and perform the multiple reduction in one go, see [7, Section 8.3] for details.

3.3.1. Subalgebra $\mathfrak{L}_5 = \langle X_3, X_4, X_5\rangle$

We find the joint invariants of

$$X_3 = \cos y\frac{\partial}{\partial x} - \cot x\sin y\frac{\partial}{\partial y} + 0\frac{\partial}{\partial t} + 0\frac{\partial}{\partial u},$$
$$X_5 = 0\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + 0\frac{\partial}{\partial t} + 0\frac{\partial}{\partial u},$$
$$X_4 = \sin y\frac{\partial}{\partial x} + \cot x\cos y\frac{\partial}{\partial y} + 0\frac{\partial}{\partial t} + 0\frac{\partial}{\partial u}.$$

From the rank of the matrix

$$\begin{pmatrix} \cos y & -\cot x\sin y & 0 & 0\\ 0 & 1 & 0 & 0\\ \sin y & \cot x\cos y & 0 & 0\end{pmatrix},$$

we see that there will be 2 independent invariants. These joint invariants are found below.

Take $I = I(x, y, t, u)$. Solving the characteristic system

$$\frac{dx}{\cos y} = \frac{dy}{-\cot x\sin y} = \frac{dt}{0} = \frac{du}{0}$$

for $L_{X_3} I = 0$ gives the constants $\sin x\sin y$, t and u. Setting $r = \sin x\sin y$ gives $I = I(r, t, u)$ as an invariant function for X_3.

If I is also invariant under X_5 then $L_{X_5} I = 0$ or $\frac{\partial}{\partial y}(I(r, t, u)] = 0$. This implies $I = I(t, u)$ which is also an invariant for X_4.

The joint invariant $I = I(t, u)$ allows us to introduce new independent and dependent variables $\xi = t$, $V(\xi) = u$. In new variables, the PDE (1.1) reduces to $\frac{\partial^2 V}{\partial \xi^2} = 0$ which corresponds to rotationally symmetric solution of wave equation on sphere.

3.3.2. Subalgebras $\mathfrak{L}_6 = \langle X_1, X_2, X_5\rangle$ and $\mathfrak{L}_7 = \langle aX_1 + bX_2, X_3, X_4, X_5\rangle$

For $\mathfrak{L}_6$, the invariant solutions are constant. There are no new non-trivial invariant solutions corresponding to $\mathfrak{L}_7$, which can be seen by an argument similar to the above.

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