TOTALLY GEODESIC HORIZONTALLY CONFORMAL MAPS (*)

by M. T. MUSTAFA (in Trieste) (**)

SUMMARY.- We obtain a characterization of totally geodesic horizontally conformal maps by a method which arises as a consequence of the Bochner technique for harmonic morphisms. As a geometric consequence we show that the existence of a non-constant harmonic morphism ϕ from a compact Riemannian manifold M^m of non-negative Ricci curvature to a compact Riemannian manifold of non-positive scalar curvature, forces M^m either to be a global Riemannian product of integral manifolds of vertical and horizontal distributions or to be covered by a global Riemannian product.

1. Introduction

A smooth map $\phi\colon M\to N$ between Riemannian manifolds is called a harmonic morphism if it preserves germs of harmonic functions i.e. if f is a real valued harmonic function on an open set $V \subseteq N$ then the composition $f \circ \phi$ is harmonic on $\phi^{-1}(V) \subseteq M$. Due to a characterization obtained by B. Fuglede [6] and T. Ishihara [9], harmonic morphisms are precisely the harmonic maps which are horizontally (weakly) conformal.

J. Vilms in [13] carried out a study of totally geodesic maps with emphasis on totally geodesic Riemannian submersions and showed that these can be characterized as Riemannian submersions with totally geodesic fibres and integrable horizontal distribution. This paper is aimed to achieve an analogous characterization of totally geodesic horizontally conformal maps and to obtain geometric consequences of this characterization in the study of totally geodesic horizontally conformal maps. The method of proof uses , as tools, the Weitzenböck formula for harmonic morphisms developed by the author in [11] and the fundamental equations of horizontally conformal submersions studied by S. Gudmundsson in [7].

(*) Pervenuto in Redazione il.

(**)Indirizzo dell'Autore: Math. Section, I.C.T.P., P.O.box 586, 34100 Trieste.
Email: mustafa@ictp.trieste.it.

The remaining part of this section presents a brief introduction to harmonic morphisms, the Bochner technique for harmonic morphisms and the fundamental equations of horizontally conformal submersions.

1.1. **Harmonic morphisms.** Recall that a map $\phi\colon M^m \to N^n$ is *harmonic* if and only if its tension field $\tau(\phi) = trace\nabla d\phi$ vanishes. The reader is referred to [2], [3] and [4] for a comprehensive account of harmonic maps.

Definition 1.1. A map $\phi\colon M^m \to N^n$ between Riemannian manifolds is called a *harmonic morphism* if $f \circ \phi$ is a real valued harmonic function on $\phi^{-1}(V) \subseteq M$ for every real valued function f which is harmonic on an open subset V of N with $\phi^{-1}(V)$ non-empty.

For a smooth map $\phi\colon M^m \to N^n$, let $C_\phi = \{x \in M \mid \text{rank}\, d\phi_x < n\}$ be its *critical set.* The points of the set $M \setminus C_\phi$ are called *regular points.* For each $x \in M \setminus C_\phi$, the *vertical space* $T_x^V M$ at x is defined by $T_x^V M = \text{Ker}\ d\phi_x$. The *horizontal space* $T_x^H M$ at x is given by the orthogonal complement of $T_x^V M$ in $T_x M$ so that $T_x M = T_x^V M \oplus T_x^H M$.

Definition 1.2. A smooth map $\phi\colon (M^m, \langle\cdot,\cdot\rangle^M) \to (N^n, \langle\cdot,\cdot\rangle^N)$ is called *horizontally (weakly) conformal* if $d\phi = 0$ on C_ϕ and the restriction of ϕ to $M \setminus C_\phi$ is a conformal submersion, that is, for each $x \in M \setminus C_\phi$, the differential $d\phi_x : T_x^H M \to T_{\phi(x)} N$ is conformal and surjective. This means that there exists a function $\lambda : M \setminus C_\phi \to \mathbb{R}^+$ such that

$$\langle d\phi(X), d\phi(Y)\rangle^N = \lambda^2 \langle X, Y\rangle^M \quad \forall X, Y \in T^H M.$$

By setting $\lambda = 0$ on C_ϕ, we can extend $\lambda : M \to \mathbb{R}_0^+$ to a continuous function on M such that λ^2 is smooth, in fact $\lambda^2 = \|d\phi\|^2/n$. The function $\lambda : M \to \mathbb{R}_0^+$ is called the *dilation* of the map ϕ. Harmonic morphisms can be characterized as follows.

Theorem 1.3 ([6], [9])**.** *A map $\phi\colon (M^m, \langle\cdot,\cdot\rangle^M) \to (N^m, \langle\cdot,\cdot\rangle^N)$ is a harmonic morphism if and only if it is a harmonic and horizontally conformal map.*

We refer the reader to [1, 6, 14] for an introduction and basic properties of harmonic morphisms. For an updated list of harmonic morphisms bibliography, see [8].

In [11], the author developed a Bochner technique for harmonic morphisms and obtained the following Weitzenböck formula for harmonic morphisms between Riemannian manifolds.

Proposition 1.4. *Let M^m and N^n be Riemannian manifolds. Let $\phi : M^m \to N^n$ be a harmonic morphism with dilation λ. Then*

$$\frac{n}{2}\Delta\lambda^2 = -\|\nabla d\phi\|^2 + \lambda^4 \mathbf{Scal^N} - \lambda^2 \mathbf{Scal^M}|_{\mathbf{H}} \tag{1.1}$$

where

$$\mathbf{Scal^M}|_{\mathbf{H}} = \sum_{s=1}^{n} \mathbf{Ricci}(e_s, e_s),$$

Δ denotes the Hodge-deRham Laplacian on functions on M and $(e_s)_{s=1}^n$, $(e_s)_{s=n+1}^m$ are orthonormal bases of $T_x^H M$ and $T_x^V M$ respectively, so that $(e_s)_{s=1}^m$ is an orthonormal basis of $T_x M = T_x^V M \oplus T_x^H M$.

1.2. **Fundamental equations of horizontally conformal submersions.** The fundamental equations of horizontally conformal submersions were established by S. Gudmundsson in [7] as a generalization of the fundamental equations of submersions found by O'Neill [12].

We recall that the fundamental tensors T, A of a submersion are defined as

$$T_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F$$

$$A_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F$$

where E,F are vector fields on M^m and $\mathcal{H}$, $\mathcal{V}$denote the orthogonal projections on the horizontal and vertical spaces respectively.

Notice that T restricted to vertical vector fields gives the second fundamental form of the fibres of the submersion and it can be easily seen that $T = 0$ is equivalent to the condition that the fibres are totally geodesic submanifolds.

For a horizontally conformal submersion it was shown that the following relation holds.

Proposition 1.5. *Let $\phi : (M^m, \langle\cdot,\cdot\rangle^M) \to (N^n, \langle\cdot,\cdot\rangle^N)$ be a horizontally conformal submersion with dilation λ and X, Y be horizontal vectors, then*

$$A_X Y = \frac{1}{2}\{\mathcal{V}[X,Y] - \lambda^2 grad_{\mathcal{V}}(\frac{1}{\lambda^2})\}.$$

From the above proposition it is obvious that in case of vertically homothetic horizontally conformal submersion, the tensor A of the horizontally conformal submersion reduces to the tensor A of a Riemannian submersion i.e. it becomes the integrability tensor of the horizontal distribution. On the other hand we see that the vanishing of A identically, implies that the horizontally conformal submersion is vertically homothetic.

Here we only state the curvature equations relevant to our work and refer the reader to [7] for other fundamental curvature equations of horizontally conformal submersions.

Lemma 1.6. *Let $m > n\geq 2$ and $\phi : (M^m, \langle\cdot,\cdot\rangle^M)\to(N^n, \langle\cdot,\cdot\rangle^N)$ be a horizontally conformal submersion, with dilation $\lambda : M\to\mathbf{R}^+$. If X, Y are horizontal and U, V are vertical vectors, then*

$$\begin{aligned}
\langle \mathbf{R^M}(X,U)X,U\rangle^M &= \langle(\nabla_U A)_X X, U\rangle^M + \langle A_X U, A_X U\rangle^M \\
&- \langle(\nabla_X T)_U X, U\rangle^M - \langle T_U X, T_U X\rangle^M \\
&+ \lambda^2\langle A_X X, U\rangle^M\langle U, grad_{\mathcal{V}}(\frac{1}{\lambda^2})\rangle^M. \\
\langle \mathbf{R^M}(X,Y)X,Y\rangle^M &= \frac{1}{\lambda^2}\langle \mathbf{R^N}(\tilde{X},\tilde{Y})\tilde{Y},\tilde{X}\rangle^N - \frac{3}{4}\|\mathcal{V}[X,Y]\|^2 \\
&+ \frac{\lambda^2}{2}[\langle X,Y\rangle^M\langle\nabla_Y grad\frac{1}{\lambda^2}, X\rangle^M - \langle Y,Y\rangle^M\langle\nabla_X grad\frac{1}{\lambda^2}, X\rangle^M \\
&+ \langle Y,X\rangle^M\langle\nabla_X grad\frac{1}{\lambda^2}, Y\rangle^M - \langle X,X\rangle^M\langle\nabla_Y grad\frac{1}{\lambda^2}, Y\rangle^M] \\
&+ \frac{\lambda^4}{4}[\|X\wedge Y\|^2\|grad(\frac{1}{\lambda^2})\|^2 + \|X(\frac{1}{\lambda^2})Y - Y(\frac{1}{\lambda^2})X\|^2].
\end{aligned}$$

2. Totally geodesic horizontally conformal maps

Definition 2.1. A map ϕ: $M^m\to N^n$ is *totally geodesic* if and only if its second fundamental form $\nabla d\phi$ vanishes, where

$$\nabla d\phi(X,Y) = (\nabla_X d\phi)Y = \nabla_X^{\phi^{-1}TN} d\phi\cdot Y - d\phi(\nabla_X^M Y)$$

for $X, Y \in \mathcal{C}(TM)$.

These maps are characterized as the maps which take geodesics of M^m linearly to geodesics of N^n.

Lemma 2.2. *A totally geodesic map has constant rank and constant energy density $e(\phi)$, where $e(\phi) = \frac{1}{2}\|d\phi\|^2$. In particular, a totally geodesic horizontally conformal map has constant dilation.*

Proof. cf. [5, page-15]. □

In this section we develop a relation between the second fundamental form and the fundamental tensors A and T of a horizontally conformal submersion, in order to achieve the following characterization of totally geodesic horizontally conformal maps between Riemannian manifolds.

Theorem 2.3. (Characterization of totally geodesic horizontally conformal maps) *Let $m > n \geq 2$ and $\phi : (M^m, \langle\cdot,\cdot\rangle^M) \to (N^n, \langle\cdot,\cdot\rangle^N)$ be a horizontally conformal map. Then ϕ is totally geodesic if and only if ϕ has constant dilation, totally geodesic fibres and integrable horizontal distribution.*

Before proving Theorem 2.3 we will prove a few results, needed in the proof of Theorem 2.3. A necessary curvature relation between $\mathbf{Scal^M}|_{\mathbf{H}}$ and $\mathbf{Scal^N}$, for a horizontally conformal map ϕ: $M^m \to N^n$, is given by the following.

Proposition 2.4. *Let $m > n \geq 2$ and M^m,N^n be Riemannian manifolds. Let $\phi : (M^m, \langle\cdot,\cdot\rangle^M) \to (N^n, \langle\cdot,\cdot\rangle^N)$ be a horizontally conformal submersion, with dilation $\lambda : M \to \mathbf{R}^+$. Then*

$$\begin{aligned}
\mathbf{Scal^M}|_{\mathbf{H}} &= \lambda^2 \mathbf{Scal^N} + \sum_{s,t=1}^{n} \{ -\frac{3}{4} \|\mathcal{V}[e_t, e_s]\|^2 \\
&+ \frac{\lambda^2}{2}[-\langle \nabla_{e_t} grad(\frac{1}{\lambda^2}), e_t\rangle^M - \langle \nabla_{e_s} grad(\frac{1}{\lambda^2}), e_s\rangle^M] \\
&+ \frac{\lambda^4}{4}[\|grad(\frac{1}{\lambda^2})\|^2 + \|e_t(\frac{1}{\lambda^2}) - e_s(\frac{1}{\lambda^2})\|^2]\} \\
&+ \sum_{s=1}^{n}\sum_{t=n+1}^{m} \{\|A_{e_s} e_t\|^2 - \langle (\nabla_{e_s} T)_{e_t} e_s, e_t\rangle^M - \|T_{e_t} e_s\|^2\} \\
&- \sum_{t=n+1}^{m} \{\frac{\lambda^2}{2} \langle \nabla_{e_t} grad_{\mathcal{V}}(\frac{1}{\lambda^2}), e_t\rangle^M + \frac{\lambda^2}{2}[\langle grad_{\mathcal{V}}(\frac{1}{\lambda^2}), e_t\rangle^M]^2\}.
\end{aligned}$$

where

$$\mathbf{Scal^M}|_{\mathbf{H}} = \sum_{s=1}^{n} \mathbf{Ricci}(e_s, e_s)$$

for an orthonormal basis $(e_s)_{s=1}^m$ of T_xM such that $(e_s)_{s=1}^n$, $(e_s)_{s=n+1}^m$ are orthonormal basis of $T_x^H M$ and $T_x^V M$ respectively.

Proof. We can write $\mathbf{Scal^M}|_{\mathbf{H}}$ as

$$\begin{aligned}
\mathbf{Scal^M}|_{\mathbf{H}} &= \sum_{s=1}^{n} \mathbf{Ricci}(e_s, e_s) \\
&= \sum_{s,t=1}^{n} \langle R^M(e_t, e_s)e_t, e_s\rangle^M + \sum_{s,=1}^{n}\sum_{t=n+1}^{m} \langle R^M(e_t, e_s)e_t, e_s\rangle^M \qquad (2.1)
\end{aligned}$$

Computing the right hand side using Lemma 1.6 gives the required relation. □

The following result on totally geodesic submersion will be needed later.

Lemma 2.5. *Let $\phi : (M^m, \langle\cdot,\cdot\rangle^M) \to (N^n, \langle\cdot,\cdot\rangle^N)$ be a submersion. If ϕ is totally geodesic then the fibres are totally geodesic.*

Proof. Let $F_x = \phi^{-1}(\phi(x))$ be the fibre at $x \in M$. Let $i : F_x \to M$ be the inclusion map, then $\phi \circ i$ is constant. Therefore, for $U, V \in T_x^V M$

$$\nabla d(\phi \circ i)(U, V) = 0$$

$$\Rightarrow \quad d\phi \cdot \nabla di(U, V) = -\nabla d\phi(diU, diV) = 0$$

$$\Rightarrow \quad \nabla di(U, V) = 0$$

Hence the fibres are totally geodesic. □

The integrability of the horizontal distribution of a totally geodesic horizontally conformal map is achieved by combining Proposition 2.4 and the Weitzenböck formula for harmonic morphisms. Precisely, we have

Lemma 2.6. *Let $m > n \geq 2$ and $\phi : (M^m, \langle\cdot,\cdot\rangle^M) \to (N^n, \langle\cdot,\cdot\rangle^N)$ be a totally geodesic horizontally conformal map. Then the horizontal distribution is integrable.*

Proof. Knowing that a totally geodesic horizontally conformal map has totally geodesic fibres and constant dilation, we have from Proposition 2.4

$$\mathbf{Scal^M}|_{\mathbf{H}} = \lambda^2 \mathbf{Scal^N} - \frac{3}{4}\sum_{s,t=1}^{n} \|\mathcal{V}[e_t, e_s]\|^2 + \sum_{s=1}^{n}\sum_{t=n+1}^{m} \|A_{e_s} e_t\|^2.$$

Using in Weitzenböck formula for harmonic morphisms we have

$$\frac{3}{4}\sum_{s,t=1}^{n} \|\mathcal{V}[e_t, e_s]\|^2 = \sum_{s=1}^{n}\sum_{t=n+1}^{m} \|A_{e_s} e_t\|^2$$

or we can write as

$$3\sum_{s,t=1}^{n} \|A_{e_t} e_s\|^2 = \sum_{s=1}^{n}\sum_{t=n+1}^{m} \|A_{e_s} e_t\|^2. \tag{2.2}$$

But we know that

$$\sum_{s,t=1}^{n} \|A_{e_t} e_s\|^2 = \sum_{s=1}^{n}\sum_{t=n+1}^{m} \|A_{e_s} e_t\|^2 \tag{2.3}$$

as follows.

$$\begin{aligned}\sum_{s,t=1}^{n}\|A_{e_t}e_s\|^2 &= \sum_{s,t=1}^{n}\sum_{u=n+1}^{m}\langle A_{e_t}e_s, e_u\rangle\langle A_{e_t}e_s, e_u\rangle \\ &= \sum_{s,t=1}^{n}\sum_{u=n+1}^{m}\langle A_{e_t}e_u, e_s\rangle\langle A_{e_t}e_u, e_s\rangle \\ &= \sum_{t=1}^{n}\sum_{u=n+1}^{m}\|A_{e_t}e_u\|^2.\end{aligned}$$

Comparing Equations 2.2 and 2.3, we have

$$\sum_{s,t=1}^{n}\|A_{e_t}e_s\|^2 = 0.$$

Hence, the horizontal distribution is integrable. □

Proof of Theorem 2.3:

$\Rightarrow$

Follows from Lemma 2.5, Lemma 2.6 and Lemma 2.2.

$\Leftarrow$

If ϕ has totally geodesic fibres, integrable horizontal distribution and constant dilation then it follows from Proposition 2.4 that

$$\mathbf{Scal^M}|_{\mathbf{H}} = \lambda^2\mathbf{Scal^N} \tag{2.4}$$

Substituting in Weitzenböck formula for harmonic morphism we obtain $\nabla d\phi = 0$.

Having seen that a totally geodesic horizontally conformal map has integrable horizontal distribution, we consider the horizontal foliation on M^m and obtain the geometric consequences of the above characterization on the horizontal and vertical foliations.

Theorem 2.7. *Let $\phi : (M^m, \langle\cdot,\cdot\rangle^M) \to (N^n, \langle\cdot,\cdot\rangle^N)$ be a totally geodesic horizontally conformal map between Riemannian manifolds with $m > n \geq 2$. Then*

(1) *The horizontal foliation is totally geodesic in M^m.*
(2) *The vertical foliation is Riemannian with bundle like metric.*

Proof.

(1) From Theorem 2.3, the horizontal distribution is integrable and the dilation λ is constant, therefore, we have a horizontal foliation on M^m which is totally geodesic as follows.

Let X, Y be horizontal vectors, then Proposition 1.5 implies that

$$\mathcal{V}\nabla^M{}_X Y = A_X Y = 0. \tag{2.5}$$

Let α_L denote the second fundamental form of a leaf L of the horizontal foliation, then

$$\alpha_L(X, Y) = \mathcal{V}\nabla^M{}_X Y = 0. \tag{2.6}$$

Hence every leaf of horizontal foliation is a totally geodesic submanifold of M.

(2) It's shown in [14] that (1)⇒(2).

□

Remark 2.8. Since a totally geodesic horizontally conformal map has totally geodesic fibres. Therefore, switching over the role of vertical and horizontal foliations in Theorem 2.7 we observe that in case of a totally geodesic horizontally conformal map, the horizontal foliation is also Riemannian. Hence, we conclude that a totally geodesic horizontally conformal map gives rise to orthogonal foliations on M^m, namely horizontal and vertical, which are Riemannian with totally geodesic leaves.

A further application of Theorem 2.3 yields the following result regarding existence of totally geodesic harmonic morphisms.

Lemma 2.9. *A harmonic morphism $\phi : M^m \rightarrow N^n$ is totally geodesic if and only if* $\ker d\phi$ *is holonomy invariant and ϕ has constant dilation.*

Proof. The only if part follows from Lemma 2.2 and [13, page-74].

Conversely suppose that $\ker d\phi$ is holonomy invariant. Then the horizontal and vertical distributions are integrable with totally geodesic integral manifolds cf. [10, pages-181,182] . The result then follows from Theorem 2.3. □

The above analysis combined with the applications of the Bochner technique presented in [11] lead to the following decomposition result for a compact Riemannian manifold of non-negative Ricci curvature, admitting a harmonic morphism.

Theorem 2.10. *Let $(M^m, \langle \cdot, \cdot \rangle^M)$ be a compact Riemannian manifold with* $\mathbf{Ricci^M} \geq 0$ *and $(N^n, \langle \cdot, \cdot \rangle^N)$ be a Riemannian manifold with* $\mathbf{Scal^N} \leq 0$. *Let ϕ be a harmonic morphism from M^m to N^n. Then either M^m is a global Riemannian product or it is covered by a global Riemannian product.*

Proof. By [11, Theorem 2.5] ϕ is totally geodesic. If M^m is simply-connected then proof follows from Theorem 2.7, Lemma 2.9 and [10, page-187].

Suppose M^m is non-simply-connected. Let $\tilde{M}$ be its universal covering space. Since ϕ can be lifted to a totally geodesic horizontally conformal map $\tilde{\phi}$ from $\tilde{M}$, therefore, the proof follows by repeating the above argument. □

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