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Wavelet optimized finite difference method with non-static re-gridding

Azad A. Siddiqui [a,1], M.T. Mustafa [b,*]

[a] National University of Sciences and Technology, EME College, Rawalpindi, Pakistan
[b] Department of Mathematical Sciences, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

Abstract

The main aim of wavelet-based numerical methods for solving partial differential equations is to develop adaptive schemes, in order to achieve accuracy and computational efficiency. The wavelet optimized finite difference method (WOFD) uses wavelets to generate appropriate grids to apply finite difference method. Its standard implementation carries out static-re-griddings after a fixed number of time steps. We present an effective implementation of WOFD method that reduces the number of static-re-griddings hence leading to reduction of FLOPS, without significant loss of accuracy. Numerical experiments are performed on different cases of heat and Schrödinger equations.

Keywords: Finite difference methods; Numerical solutions; Partial differential equations; Wavelet-based methods

1. Introduction and description of the method

The mathematical modeling of most of the physical processes leads to partial differential equations of the form, for which the methods of finding analytic solutions are either not known or hard to develop. Therefore, one looks for their numerical solutions. Many of these partial differential equations exhibit localized high frequency behavior. If the conventional finite difference approach is employed for numerical solution of such problems, then one requires a very fine grid to incorporate the steep gradients. The disadvantage is that a very large number of grid points are required, which heavily increases the computational cost. Furthermore, most of these grid points are located in regions where the solution is quite smooth and such a fine gridding is not required there, for the desired accuracy. Therefore, it is appropriate to employ adaptive methods; for instance, adaptive finite difference, adaptive finite element or using wavelets.

Wavelets provide a natural tool to deal with such problems. Several wavelet-based methods are available in literature to solve partial differential equations. These include filter bank approach [9], interpolating wavelet methods [4,5], Galerkin type methods [1,2], collocation method [3] and WOFD method [6–8].

* Corresponding author.
E-mail addresses: azad-eme@nust.edu.pk, azad@ifm.liu.se (A.A. Siddiqui), tmustafa@kfupm.edu.sa (M.T. Mustafa).
[1] Present address: Department of Physics and Measurement Technology, Linköping University, SE-583 81 Linköping, Sweden.

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The WOFD method is due to Jameson [6–8] and is a variant of adaptive wavelet Galerkin method. The main idea of this method is to use wavelets to generate appropriate grid only, and use the finite difference method to solve the partial differential equation. Roughly speaking, WOFD method can be divided in two parts, "*the grid generation*" and "*differencing*". Firstly, a wavelet analysis is carried out to find the local variations of the data. This provides a mechanism for grid generation [7] in such a manner that grid points are placed sparsely in smooth or low variation regions, and finer in regions of strong variations or high oscillations. Secondly, a finite difference scheme is used for calculations on this generated grid. WOFD method was further extended to an adaptive order numerical method (WOFD2) in [8]. This differs from WOFD in the sense that an appropriate order of finite difference scheme is adapted during the calculations.

In this paper, we present the idea of "*non-static re-gridding*" which provides an effective implementation of WOFD method. Hence saving the computational time involved in griddings and reducing the number of FLOPS.

The amount of work involved to solve a typical problem, on a finer grid, using conventional finite difference method is approximately same as required to generate coarser grid in WOFD method. Now using finite difference scheme on this coarser grid amounts to the extra work done in WOFD method. So if re-gridding is done at each time step then this method will prove to be expensive than the conventional finite difference method. However, the real advantage of WOFD method over finite difference method is that, for the desired accuracy, re-griddings are not required at each time step. In standard implementation of WOFD method, re-griddings are done after a fixed number of time steps, which we call as "*static re-gridding*".

The aim of non-static re-griddings is to reduce the total number of re-griddings (without significantly loosing the accuracy) involved in the process of finding the solution, as this is the key to improve the time-complexity of the WOFD method. Instead of doing static re-griddings, we analyze the solution after a fixed number of time steps, to see if re-gridding is needed or not. Suppose in a standard implementation of WOFD method the static re-griddings are performed N times (after the first gridding at initial time). Let $T_i (i = 1, 2, \ldots, N)$ denote the time at the ith static re-gridding and δt be the time step size in the finite difference scheme. Then our test (to check whether or not ith re-gridding is needed) basically involves comparison of solution u_i at time $T_i - \delta t$ with the solution u_{i-1} at time $T_{i-1} - \delta t$, for $i = 2, 3, \ldots, N$. For $i = 1$ the solution at $T_1 - \delta t$ is compared with the initial condition. Precisely the l_∞ norm of the difference, $(u_i - u_{i-1})$, of the solutions being compared is considered for re-gridding. If this value is greater than a given threshold, Δs, then re-gridding is done otherwise it is skipped. The number of FLOPS involved in this comparison is negligible as compared to the FLOPS involved in doing the re-gridding. The non-static re-gridding reduces the number of re-griddings significantly for ranges which have low or smooth variations in time, even if sharp spatial variations are present. In the presence of oscillatory behavior, in time, our method does not skip re-griddings significantly, showing the reliability.

In Sections 2 and 3 we present our numerical experiments on heat equation and Schrödinger equation respectively, using WOFD with non-static re-gridding. The comparison with corresponding solutions obtained using static re-griddings is also presented.

2. Heat equation

The first example considered is

$$
\begin{aligned}
&u_t = u_{xx} + 4\pi^2 \sin(2\pi x), \quad 0 < x < 1,\ 0 < t \leqslant 1, \qquad (1)\\
&u(0,t) = u(1,t) = 1, \quad t \geqslant 0,\\
&u(x,0) = 1.
\end{aligned}
$$

The exact solution is $u(x,t) = 1 + (1 - e^{-4\pi^2 t}) \sin(2\pi x)$, which does not have oscillatory variation in time (Fig. 1).

In the following sub-sections we present numerical solutions of Eq. (1), using non-static re-griddings. We have obtained results for two different initial grid sizes in order to see the consistency of the average grid points involved in the calculations.

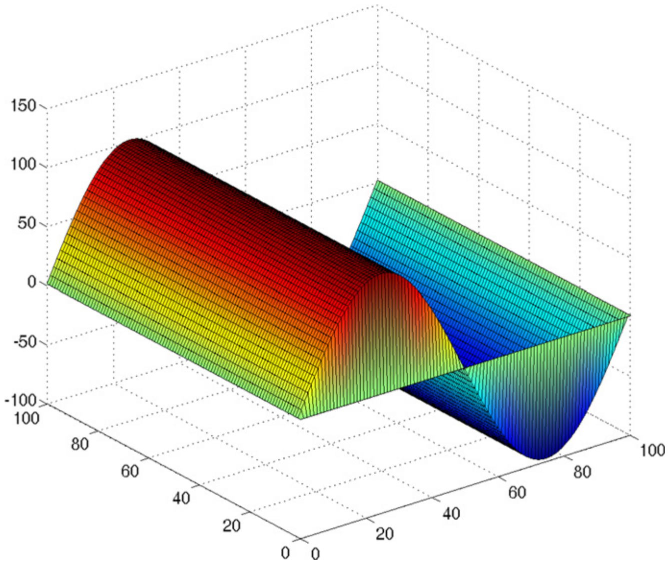

Fig. 1. Plot of the analytic solution.

2.1. Initial grid size of 512 points with 32 static re-griddings

The numerical solution of the above problem using WOFD with 32 static re-griddings involves:

- Mega FLOPS for griddings = 4.22;
- average grid size (number of points in finite difference calculations) = 35

and has an error 1.2077×10^{-8}, where error means the difference from the analytic solution. For comparison of results, details of solutions obtained using non-static re-griddings for different values of our threshold parameter Δs are given in the table below:

Δs	Number of griddings	Mega FLOPS (for griddings)	Average grid size (points)	Error
0.2	2	0.99	72	2.0114×10^{-8}
0.01	5	1.29	51	1.9697×10^{-8}
0.003	6	1.34	35	1.2077×10^{-8}
0.001	7	1.45	35	1.2077×10^{-8}
10^{-7}	14	2.23	35	1.2077×10^{-8}
10^{-10}	20	2.29	35	1.2077×10^{-8}
10^{-15}	28	3.78	35	1.2077×10^{-8}

Figs. 2 and 3 give the solution for $\Delta s = 0.003$, which is the optimal solution both in terms of FLOPS (without any loss of accuracy) and corresponds to the smallest average grid size.

2.2. Initial grid size of 256 points with 32 static re-griddings

The numerical solution of the above problem using WOFD with 32 static re-griddings involves:

- Mega FLOPS for griddings = 2.49;
- average grid size (number of points in finite difference calculations) = 32

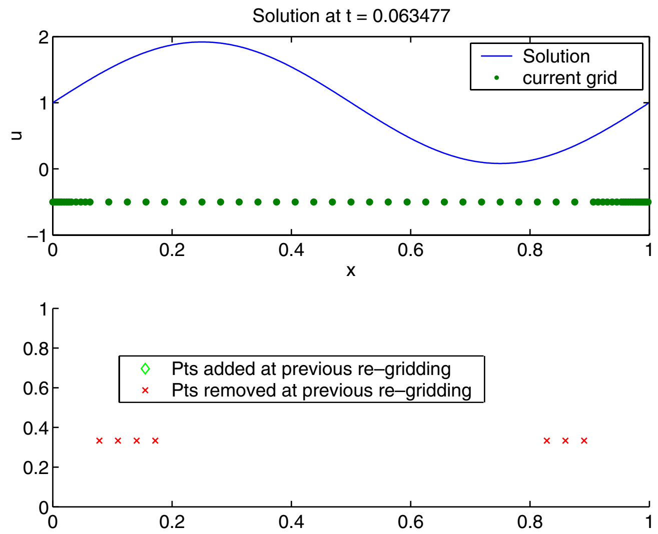

Fig. 2. Solution; added/removed points for $\Delta s = 0.003$.

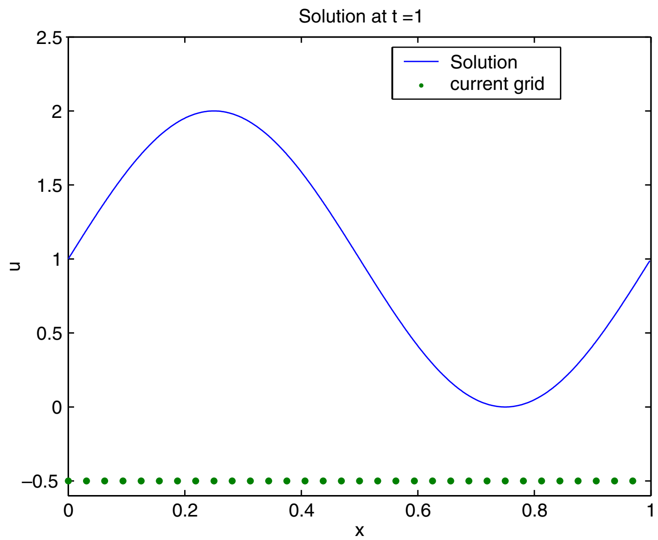

Fig. 3. Final solution for $\Delta s = 0.003$.

and has an error 1.0984×10^{-8}. The details of solutions obtained using non-static re-griddings for different values of our threshold parameter Δs are given in the following table:

Δs	Number of griddings	Mega FLOPS (for griddings)	Average grid size (points)	Error
0.5	2	0.56	61	1.1917×10^{-8}
0.1	3	0.62	32	1.0984×10^{-8}
0.05	4	0.68	32	1.0984×10^{-8}
10^{-6}	12	1.20	32	1.0984×10^{-8}
10^{-9}	18	1.59	32	1.0984×10^{-8}
10^{-14}	28	2.23	32	1.0984×10^{-8}

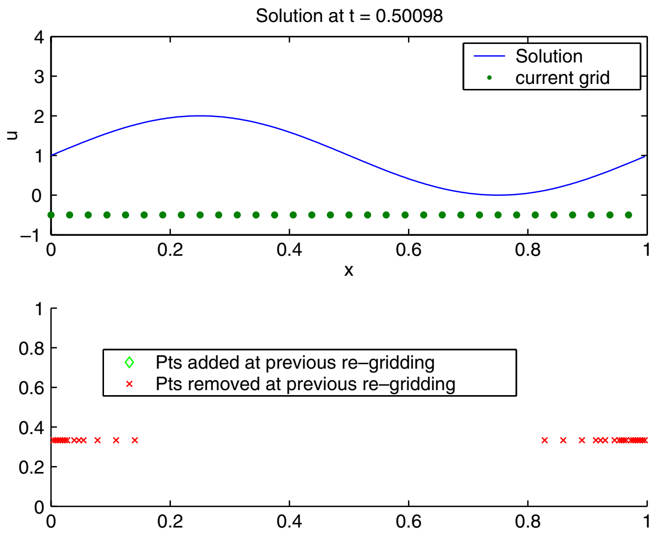

Fig. 4. Solution; added/removed points for $\Delta s = 0.1$.

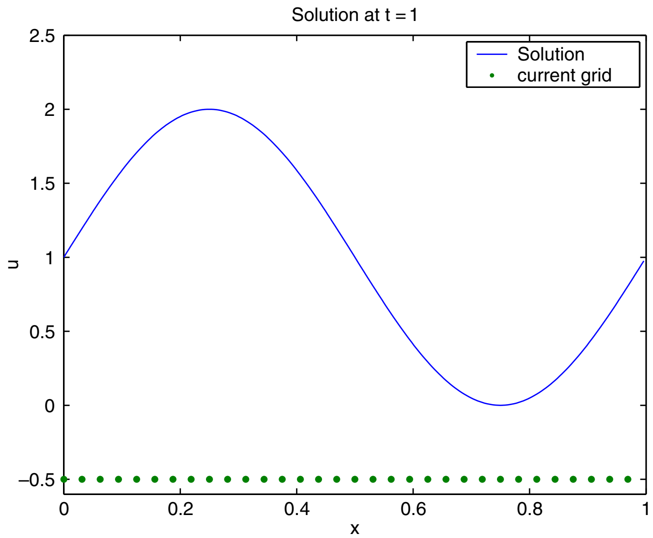

Fig. 5. Final solution for $\Delta s = 0.1$.

Figs. 4 and 5 give the optimal solution for $\Delta s = 0.1$.

3. Schrödinger equation

The second example considered is

$$\begin{aligned} &u_t = i(u_{xx} + 2|u|^2 u), \quad -L/2 < x < L/2, \ t > 0, \\ &u(-L/2, t) = u(L/2, t), \\ &u(x, 0) = h(x). \end{aligned} \tag{2}$$

The numerical solutions for $L = 64$ and $h(x) = 2\operatorname{sech} x$ are presented in the following sub-sections. For these cases the errors are computed by taking the difference from the numerical solution obtained using coarsest grid (with 1024 points). Here, keeping the initial grid size same, we investigate two different cases on the basis of different numbers of static re-griddings.

3.1. Initial grid size of 1024 points with 32 static re-griddings

Solution of the above problem using WOFD with 32 static re-griddings involves:

- Mega FLOPS for griddings = 27.92;
- average grid size (number of points in finite difference calculations) = 120

and has an error 4.004×10^{-8}. The details of solutions obtained using non-static re-griddings for different values of our threshold parameter Δs are given in the table below:

Difference	Number of griddings	Mega FLOPS (for griddings)	Average grid size (points)	Error
1.25	6	7.22	156	5.1672×10^{-4}
0.8	12	11.83	130	6.4576×10^{-6}
0.575	17	15.70	122	3.1162×10^{-7}
0.55	18	16.47	121	1.7094×10^{-7}
0.545	19	17.23	120	4.4721×10^{-8}
0.5	21	18.79	120	3.5408×10^{-8}
0.47	24	21.13	120	3.4610×10^{-8}
0.44	29	25.03	120	3.5986×10^{-8}

Figs. 6 and 7 give the optimal solution for $\Delta s = 0.545$.

3.2. Initial grid size of 1024 points with 64 static re-griddings

The numerical solution of the above problem using WOFD with 64 static re-griddings involves:

- Mega FLOPS for griddings = 55.65;
- average grid size (number of points in finite difference calculations) = 120

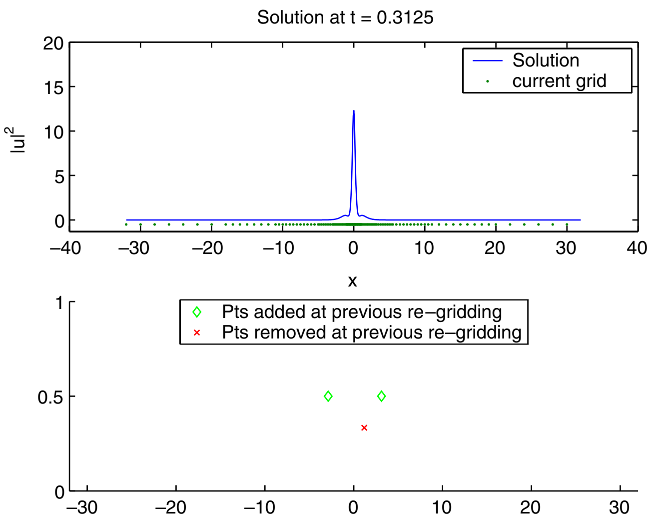

Fig. 6. Solution; added/removed points for $\Delta s = 0.545$.

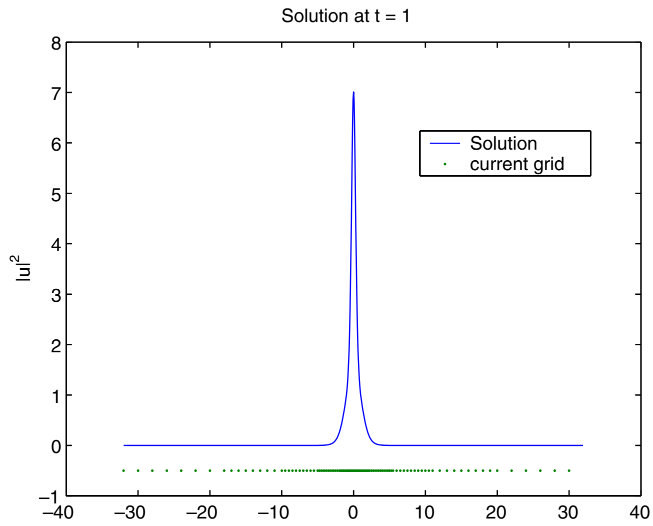

Fig. 7. Final solution for $\Delta s = 0.545$.

and has an error 3.3008×10^{-8}. The details of solutions obtained using non-static re-griddings for different values of our threshold parameter Δs are given in the following table:

Difference	Number of griddings	Mega FLOPS (for griddings)	Average grid size (points)	Error
0.65	11	14.02	178	2.0606×10^{-4}
0.5	17	18.35	146	3.3152×10^{-5}
0.38	23	22.74	129	2.9600×10^{-6}
0.33	27	25.83	124	6.0347×10^{-7}
0.295	33	30.43	123	1.4805×10^{-7}
0.27	37	33.56	120	6.7192×10^{-8}
0.25	43	38.22	120	2.8401×10^{-8}
0.23	51	44.43	120	2.7880×10^{-8}
0.22	60	51.46	120	3.3838×10^{-8}

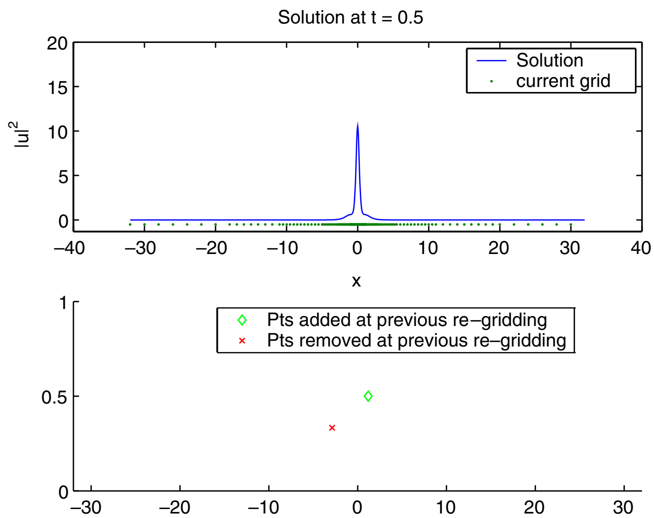

Fig. 8. Solution; added/removed points for $\Delta s = 0.27$.

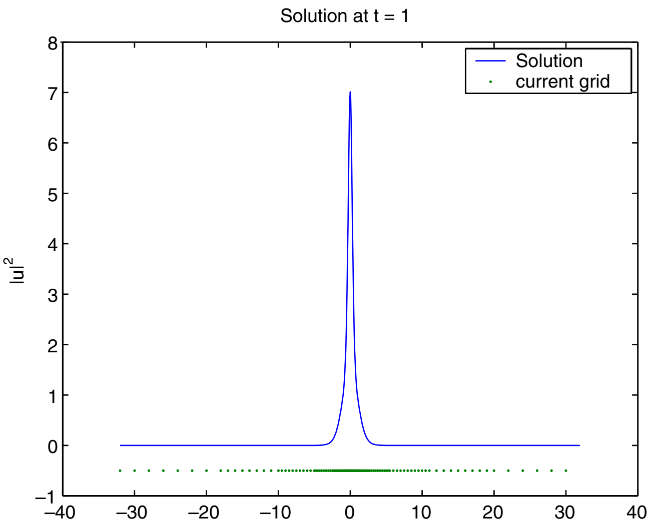

Fig. 9. Final solution for $\Delta s = 0.27$.

Figs. 8 and 9 give the optimal solution for $\Delta s = 0.27$.

4. Conclusion

The WOFD method for solving partial differential equation uses wavelets to generate appropriate grid for finite difference scheme. In standard implementation of WOFD method, re-griddings are done after a fixed number of time steps, which we call as "*static re-gridding*". In this paper we have presented the idea of "*non-static re-gridding*" in order to achieve effective implementation of WOFD method. Our scheme is centered around a threshold value Δs, which is explained in Section 1. The method is applied on Heat and Schrödinger equations and solutions are obtained for different values of Δs. An optimal value of Δs, corresponding to the optimal solution, is obtained for each case. However, these values are different even for same problem with varying initial grid size or different number of static re-griddings. It is conjectured that a common optimal threshold value can be obtained for similar problems by taking Δs to be the l_∞-norm of the relative difference (i.e. $(u_i - u_{i-1})/u_{i-1}$ in the notation of Section 1).

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