King Fahd University Of Petroleum & Minerals
Department Of Mathematics And Statistics
STAT416 : Stochastic Processes for Actuaries (181)
First Exam **Thursday October 18, 2018**
Name: **ID:**

Question Number	Full Mark	Marks Obtained
One	10	
Two	8	
Three	18	
Four	8	
Five	15	
Six	13	
Seven	13	
Eight	12	
Nine	10	
Ten	13	
Total	120	

Question.1 (**2+2+2+2+2=10-Points**) Define the following:

(a) Markov Chain:

(b) Communicate states (say state i and j) :

(c) Reducible Markov chain:

(d) Counting process $\{N(t); t \geq 0\}$:

(e) Nonhomogeneous Poisson process :

Question .2 **(4+4=8-Points)** Assume that a man's profession can be classified as professional, skilled labourer, or unskilled labourer. Assume that, of the sons of professional men, 80 percent are professional, 10 percent are skilled labourers, and 10 percent are unskilled labourers. In the case of sons of skilled labourers, 60 percent are skilled labourers, 20 percent are professional, and 20 percent are unskilled. Finally, in the case of unskilled labourers, 50 percent of the sons are unskilled labourers, and 25 percent each are in the other two categories. Assume that every man has at least one son, and form a Markov chain by following the profession of a randomly chosen son of a given family through several generations.

(a) Set up the matrix of transition probabilities $\boldsymbol{P}$

(b) Find the probability that a randomly chosen grandson of an unskilled labourer is a professional man.

Question.3 **(8+2+8=18-Points)** The MIT football teams performance in any given game is correlated to its morale. If the team has won the past two games, then it has a 0.7 probability of winning the next game. If it lost the last game but won the one before that, it has a 0.4 probability of winning. If it won its last game but lost the one before that it has a 0.6 probability of winning. Finally if it lost the last two games it has only a 0.3 probability of winning the next game. No game can end up in a draw. Consider a starting time when the team has won its preceding two games.

(a) Write the transition probability matrix for the Markov chain for the MIT football teams performance.

(b) Find the probability that the first future loss will be followed by another loss.

(c) Evaluate the steady-state probabilities or the limiting probability vector.

Question.4 **(6+2=8-Points)**

Consider the following Markov chain with states $\{0, 1, 2, 3, 4\}$ and transition probability matrix given by:

$$\boldsymbol{P} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1/4 & 1/4 & 1/3 & 0 & 1/6 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/4 & 3/4 \\ 0 & 0 & 0 & 5/6 & 1/6 \end{pmatrix}$$

(a) Identify the classes, the recurrent, transient, and absorbing states.

(b) Find the period of each recurrent stante, if exists.

Question 5. **(10+5=15-Points)**

Let $\boldsymbol{P} = \begin{pmatrix} 0.4 & 0.6 \\ 0.7 & 0.3 \end{pmatrix}$ be the transition probability matrix of a Markov chain. Given that the eigenvalues are $1, -0.3$ and the corresponding eigenvectors are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 7/6 \end{pmatrix}$

(a) Find $\boldsymbol{P}^n$

(b) If the initial vector is $(1, 0)$, find $\lim\limits_{n\to\infty} \boldsymbol{P}^n$.

Question.6 (**2+6+5=13-Points**)

Consider the gamblers' ruin problem with $p = \frac{1}{3}$, and $N = 3$. Starting with 1 unit, then

(a) Write down the transition probability matrix

(b) Determine the expected amount of time the gambler has 2units

(c) Find the probability that eventually, the gambler has a full fortune

Question 7. (**11+2=13-Points**)

Let $\beta(z) = Az^2 + Bz + C$, with $A > 0, B > 0, C > 0$ be a probability generating function of a branching process with probability of extinction $\pi_0 = \frac{3}{8}$.

(a) Find A, B, and C (**Remark** : Answer is not unique)

(b) If $\beta(z) = \frac{8}{33}z^2 + \frac{2}{3}z + \frac{1}{11}$. Find $P\left(X_1 = 1 | X_0 = 1\right)$

Question.8 (**8+4=14-Points**)

The number of failures which occurs in a computer network over the interval $(0, t]$ can be described by a Poisson process $\{N(t); t \geq 0\}$ with an average of one failure every 4 hours ($\lambda = 0.25/hr$).

(a) What is the probability of at most one failure in $(0, 8]$, and at least two failures in $(8, 16]$? (time unit=hour).

(b) Given that two failures occurred in the first 16 hours ($(0, 16]$), what is the probability that both occurred in the first 8 hours ($(0, 8]$)?

Question 9. **(4+4+2=10-Points)**

Customers arrive at an automated teller machine at the times of nonhomogeneous Poisson process with rate function given by:

$$\lambda(t) = \begin{cases} 4t & ,9:00am \leq t < 10:00am \\ 2+2t & ,10:00am \leq t < 12:00pm \\ 6 & ,12:00pm \leq t < 2:00pm \\ 25-2t & ,2:00pm \leq t \leq 4:00pm \end{cases}$$

(a) What is the probability that the total number of 8 customers arrive before 11:00am?

(b) What is the probability that 5 customers arrive between 12:00pm and 2:00pm?

(c) What is the expected number of customers during the last business hour?

Question 10. **(3+4+6=13-Points)**

An emergency department in a big hospital has three entrances, A, B, and C. Patients arrive to the department according to a Poisson process with rates $\lambda_A = 2$, $\lambda_B = 3.5$, and $\lambda_C = 5$ per minute. Arrivals to different entrances are independent. Given these information, answer the following.

(a) What is the probability that a total 10 patients arrived to the department in a 1-minute period?

(b) What is the probability that one patient arrived from entrance A and 5 patients from entrance C in a 1-minute period?

(c) What is the probability that 1 patient arrived to the department from entrance A in the interval $(0, 2]$ and a total of two patients in the interval $(1, 3]$ from the same entrance?