King Fahd University of Petroleum and Minerals
Department of Mathematics & Statistics
Math 565 Final Exam
The First Semester of 2018-2019 (181)
Time Allowed: 120mn

Name: ID number:

Textbooks are not authorized in this exam

Problem #	Marks	Maximum Marks
1		20
2		20
3		20
4		20
5		20
Total		100

Problem 1:(20pts) Consider the system

$$\frac{dx}{dt} = x^3 - y^2$$
$$\frac{dy}{dt} = y^3 - x^2.$$

Let $E(t) = x^2(t) + y^2(t)$.
1.)(8pts) Show that

$$\frac{d}{dt}E(t) \leq 3E^2(t) + E(t). \tag{1}$$

2.)(12pts)Show that: if $\int_0^t E(s)ds \leq \alpha$, $\forall t \geq 0$, then integrating (1) we find

$$E(t) \leq (E(0) + \alpha)e^{3\alpha}, \quad \forall t \geq 0.$$

Solution

1.)

$$\frac{dx}{dt} = x^3 - y^2 \quad \times x \quad \Rightarrow \frac{1}{2}\frac{d}{dt}x^2 = x^4 - y^2x$$

$$\frac{dy}{dt} = y^3 - x^2 \quad \times y \quad \Rightarrow \frac{1}{2}\frac{dy^2}{dt} = y^4 - x^2y$$

$$\frac{1}{2}\frac{d}{dt}(x^2+y^2) = x^4+y^4 - y^2x - x^2y$$
$$\leq x^4+y^4 + \frac{y^4}{2}+\frac{x^2}{2}+\frac{x^4}{2}+\frac{y^2}{2}$$
$$\leq \frac{3}{2}(x^4+y^4) + \frac{1}{2}(x^2+y^2)$$
$$\leq \frac{3}{2}(x^2+y^2)^2 + \frac{1}{2}(x^2+y^2)$$

2.)

$$\frac{d}{dt}E \leq 3E^2(t) + E(t) \quad \times e^{-3\int_0^t E(\tau)d\tau}$$

$$\Rightarrow \frac{d}{dt}\left(E(t)e^{-3\int_0^t E(\tau)d\tau}\right) \leq E(t)e^{-3\int_0^t E(\tau)d\tau}$$

We integrate between 0 and t. We find

$$E(t)e^{-3\int_0^t E(\tau)d\tau} - E(0) \leq \int_0^t E(s)e^{-3\int_0^s E(\tau)d\tau}ds$$

$$\Rightarrow \quad E(t) \leq E(0)e^{3\int_0^t E(\tau)d\tau} + \int_0^t E(s)e^{3\int_s^t E(\tau)d\tau}ds$$

$$\leq \left(E(0) + \int_0^t E(s)ds\right)e^{3\int_0^t E(\tau)d\tau}$$

$$\leq (E(0) + \alpha)e^{3\alpha}, \quad \forall t \geq 0.$$

Problem 2:(20pts) Consider the system

$$\frac{dx}{dt} = x + y - 2x(x^2+y^2)$$
$$\frac{dy}{dt} = -x + y - 3y(x^2+y^2).$$

1.)(8pts) Show that the system has no periodic solution inside the region
$R = \{(x,y) \in \mathbb{R}^2,\ 1 - 9x^2 - 11y^2 \geq 0\}$.

2.)(12pts) Show that the system has at least one closed orbit.

<u>Solution</u>

1) $\nabla \cdot F = 1 - 2(x^2+y^2) - 4x^2 + 1 - 3(x^2+y^2) - 6y^2$
$= 2 - 9x^2 - 11y^2$

If $(x,y) \in R \Rightarrow 1 - 9x^2 - 11y^2 \geq 0 \Rightarrow 2 - 9x^2 - 11y^2 \geq 1$

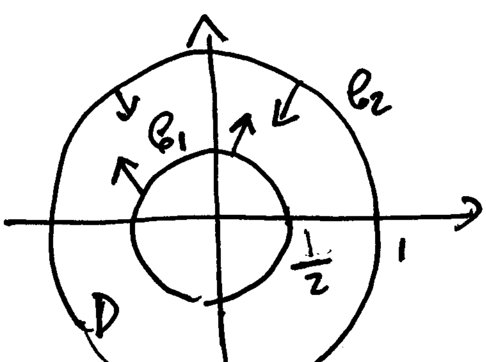
Bindixson's criteria $\Rightarrow$ there is no periodic Solution inside R.

2) $\frac{dx}{dt} = x + y - 2x(x^2+y^2) \quad \times x \Rightarrow \frac{1}{2}\frac{d}{dt}x^2 = x^2 + xy - 2x^2(x^2+y^2)$

$\frac{dy}{dt} = -x + y - 3y(x^2+y^2) \quad \times y \Rightarrow \frac{1}{2}\frac{d}{dt}y^2 = -xy + y^2 - 3y^2(x^2+y^2)$

$\frac{1}{2}\frac{d}{dt}(x^2+y^2) = (x^2+y^2)(1 - 2x^2 - 3y^2)$

Let $D = \{(x,y) \in \mathbb{R}^2 / \frac{1}{4} \leq x^2+y^2 \leq 1\}$



Let $V(x,y) = x^2+y^2$

$\dot{V} \leq 2(x^2+y^2)(1-2x^2-2y^2)$, $\dot{V}|_{C_2} < 0$

$\dot{V} \geq 2(x^2+y^2)(1-3x^2-3y^2)$, $\dot{V}|_{C_1} > 0$

D is a trapping region.

Poincaré-Bindixson's theorem
$\Rightarrow$ There is one periodic solution.

Problem 3:(20pts) Consider the system

$$\frac{dx}{dt} = y^2 - 1$$
$$\frac{dy}{dt} = x^3 - y.$$

1.)(4pts) Find all the critical points of the system.
2.)(4pts) Let $x = u + 1$ and $y = v + 1$. Write the differential equations in the variable (u, v).
3.)(12pts) Study the stability of the point $(x, y) = (1, 1)$.

Solution

1.) $\begin{cases} y^2 - 1 = 0 \\ x^3 - y = 0 \end{cases} \Rightarrow y = 1, \; -1 \quad \Downarrow \quad x = 1, \; x = -1 \quad \Rightarrow A\begin{pmatrix}1\\1\end{pmatrix}, \; B\begin{pmatrix}-1\\-1\end{pmatrix}$

2.) $x = u+1$, $y = v+1$ $\Rightarrow \begin{cases} \frac{du}{dt} = (v+1)^2 - 1 = 2v + v^2 \\ \frac{dv}{dt} = (u+1)^3 - (v+1) = u^3 + 3u^2 + 3u - v \end{cases}$

Let $F(u,v) = \begin{pmatrix} v^2 \\ u^3 + 3u^2 \end{pmatrix}$.

$$\frac{|F(u,v)|}{|u|+|v|} = \frac{v^2 + u^2|3+u|}{|u|+|v|} \leq \frac{(|u|+|v|)^2(3+|u|)}{|u|+|v|}$$
$$\leq (|u|+|v|)(3+|u|)$$

$$\lim_{|u|+|v| \to 0} \frac{|F(u,v)|}{|u|+|v|} = 0$$

$\begin{cases} \frac{du}{dt} = 2v \\ \frac{dv}{dt} = 3u - v \end{cases} \iff \dot{X} = \begin{pmatrix} 0 & 2 \\ 3 & -1 \end{pmatrix} X, \quad X\begin{pmatrix}u\\v\end{pmatrix}$

$-\lambda(-1-\lambda) - 6 = 0$, $\lambda^2 + \lambda - 6 = 0$

$\lambda = -3, \; \lambda = 2$

The critical point A is an unstable saddle point.

Problem 4:(20pts) Consider the system

$$\begin{aligned}\frac{dx}{dt} &= x + xy^2\\ \frac{dy}{dt} &= y + x^2y.\end{aligned}$$

Let $V(x,y) = x^2 + xy + y^2$ and $\frac{d}{dt}V(x(t),y(t)) = V^*(x(t),y(t))$
Show that $V^*(x,y)$ is positive definite on a domane D. Thus, (0,0) is unstable.

Solution

$$\frac{dx}{dt} = x + xy^2 \quad \times x \quad \Rightarrow \quad \frac{1}{2}\frac{d}{dt}x^2 = x^2 + x^2y^2 \qquad (1)$$

$$\frac{dy}{dt} = y + x^2y \quad \times y \quad \Rightarrow \quad \frac{1}{2}\frac{d}{dt}y^2 = y^2 + x^2y^2 \qquad (2)$$

$$\begin{aligned}\frac{d}{dt}(xy) &= xy' + yx'\\ &= x(y + x^2y) + y(x + xy^2)\\ &= 2xy + yx^3 + xy^3\end{aligned}$$

$$\Rightarrow \quad \frac{1}{2}\frac{d}{dt}(xy) = xy + \frac{1}{2}yx^3 + \frac{1}{2}xy^3 \qquad (3)$$

$$(1) + (2) + (3) \Rightarrow \frac{1}{2}\frac{d}{dt}V = \underbrace{x^2 + y^2 + 2x^2y^2 + xy + \frac{1}{2}yx^3 + \frac{1}{2}xy^3}_{\frac{1}{2}V^*}$$

$$V^*(x,y) = 2x^2 + 2y^2 + 4x^2y^2 + 2xy + yx^3 + xy^3$$

$$\left.\begin{aligned}V^*_x &= 4x + 8xy^2 + 2y + 3x^2y + y^3\\ V^*_y &= 4y + 8x^2y + 2x + x^3 + 3xy^2\end{aligned}\right\} \nabla V^*/(0,0) = 0$$

$$\left.\begin{aligned}V^*_{xx} &= 4 + 8y^2 + 6xy\\ V^*_{yy} &= 4 + 8x^2 + 6xy\\ V^*_{xy} &= 16xy + 2 + 3x^2 + 3y^2\end{aligned}\right\} \left. V^*_{xx}V^*_{yy} - V_{xy}^2 \right|_{(0,0)} = 16 - 2 > 0$$

$$V^*_{xx}(0,0) = 4 > 0$$

$\Rightarrow$ V^* has a local minimum at (0,0)
V^* is positive definite on some D.
V is positive definite on $\mathbb{R}^2$ (5) $\Rightarrow$ (0,0) is unstable.

Problem 5:(20pts) Consider the system

$$\frac{dx}{dt} = -x + g(y)$$
$$\frac{dy}{dt} = -y + h(x).$$

Assume that, for all $s \in \mathbb{R}$, we have

$$|g(s)| \leq \frac{|s|}{2}, \quad |h(s)| \leq \frac{|s|}{2}.$$

Show that the critical point $(0,0)$ is globally asymptotically stable.

<u>Solution</u>

$$\frac{dx}{dt} = -x + g(y) \quad \times x \Rightarrow \frac{1}{2}\frac{d}{dt}x^2 = -x^2 + x\,g(y)$$

$$\frac{dy}{dt} = -y + h(x) \quad \times y \Rightarrow \frac{1}{2}\frac{d}{dt}y^2 = -y^2 + y\,h(x)$$

$$\frac{1}{2}\frac{d}{dt}\underbrace{(x^2+y^2)}_{V} = \underbrace{-x^2-y^2+x\,g(y)+y\,h(x)}_{\frac{1}{2}V^*}$$

$$-V^* = x^2+y^2 - x\,g(y) - y\,h(x)$$

$$|x\,g(y)| \leq \frac{|x|\,|y|}{2} \leq \frac{x^2}{4}+\frac{y^2}{4}$$

$$-x\,g(y) > -\left(\frac{x^2}{4}+\frac{y^2}{4}\right)$$

$$|y\,h(x)| \leq \frac{|y|\,|x|}{2} \leq \frac{x^2}{4}+\frac{y^2}{4}$$

$$-y\,h(x) \geq -\left(\frac{x^2}{4}+\frac{y^2}{4}\right)$$

Thus, $-V^* \geq x^2+y^2 - \left(\frac{x^2}{2}+\frac{x^2}{2}\right) = \frac{x^2+y^2}{2}$.

So, V is positive definite on $\mathbb{R}^2$

V^* is negative definite on $\mathbb{R}^2$

$\longrightarrow$ $(0,0)$ is globally asymptotically stable.