King Fahd University of Petroleum & Minerals
Department of Mathematics and Statistics

Math 550 – Linear Algebra (Term 181)

Final

Notes:

- Duration = **3 hours.**
- Problem 1 is worth **20 points.**
- Problems 2-6 are worth **10 points** each.

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(1) Let A be the following matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and let T be the linear operator on $\mathbb{R}^4$ associated to A in the standard basis $S := \{e_1, e_2, e_3, e_4\}$.

(a) [4 points] Use row and column operations on $\mathbb{R}[x]^{4\times 4}$ to reduce $xI - A$ to its Smith normal form.

(b) [3 points] Use (a) to give the invariant factors, the minimal and characteristic polynomials of T.

(c) [4 points] Give an explicit cyclic decomposition of $\mathbb{R}^4$ under T.

(d) [3 points] Give the Jordan form for T.

(e) [4 points] Give the primary decomposition of $\mathbb{R}^4$ under T.

(f) [2 points] For each i, let E_i denote the projection of $\mathbb{R}^4$ on the primary component W_i. For each i, give $[E_i]_S$, the matrix of E_i relative to S.

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(2) **(a)** [5 points] In the $\mathbb{R}$-vector space of polynomials of degree at most 3, equipped with the inner product $(f|g) = \int_0^1 f(t)g(t)dt$, let W denote the subspace of scalar polynomials. Give an explicit basis for the orthogonal complement of W.

(b) [5 points] In the real inner product space of real-valued continuous functions on the interval $[-1, 1]$, endowed with the inner product $(f|g) = \int_{-1}^1 f(t)g(t)dt$, find the orthogonal complement of the subspace of odd functions.

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(3) Let V be a finite-dimensional complex inner product space and let T be a linear operator on V. Prove that T is self-adjoint if and only if $(T\alpha|\alpha)$ is real for every $\alpha \in V$.

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(4) Let V be a finite-dimensional complex inner product space. Let $S \coloneqq \{\alpha_1, \dots, \alpha_n\}$ be an orthonormal basis for V and let T be a normal linear operator on V such that $A \coloneqq [T]_S$ is upper-triangular with entries a_{ij} (that is, $a_{ij} = 0 \ \forall\, i > j$).

(a) [4 points] Prove that $a_{1j} = 0 \ \forall\, j \geq 2$.

(b) [4 points] Use induction on i to prove that $a_{ij} = 0 \ \forall\, j \geq i + 1$.

(c) [2 points] Deduce from above that: "*a linear operator T on V is normal if and only if V has an orthonormal basis consisting of characteristic vectors for T.*"

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(5) Let f and g be two bilinear forms on a finite-dimensional vector space V. Suppose that f is non-degenerate.

(a) [6 points] Prove that g is non-degenerate if and only if there exists $T \in End(V)$ with $f(\alpha, \beta) = g(\alpha, T\beta)$ for all $\alpha, \beta \in V$.

(b) [4 points] Is the result given in (a) true if f is singular? [Justify]

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(6) Let V be a finite-dimensional vector space over a field F and let f be a non-degenerate symmetric bilinear form on V.

(a) [8 points] Let $B \coloneqq \{\alpha_1, \dots, \alpha_n\}$ be a basis for V. Use the linear transformation (of Section 10.1) $L_f: V \longrightarrow V^*$ to show that there exists a unique basis $B' \coloneqq \{\alpha'_1, \dots, \alpha'_n\}$ of V such that:

- $f(\alpha_i, \alpha'_j) = \delta_{ij}$ for all i and j, and hence
- $\alpha = \sum_i f(\alpha, \alpha'_i)\alpha_i = \sum_i f(\alpha_i, \alpha)\alpha'_i$ for every $\alpha \in V$.

(b) [2 points] Show that if $F = \mathbb{C}$, then there is a basis B of V such that $B' = B$.

------------------------------ Good Luck -----------------------------------