King Fahd University of Petroleum & Minerals
Department of Mathematics and Statistics

Math 550 - Linear Algebra (Term 181)

Exam 2

Notes:
- Duration = **3 hours.**
- Each problem is worth **10 points.**

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(1) Let V denote the vector space of $n \times n$ matrices over a field F.
For each $A \in V$, let $T_A \in End(V)$ defined by $T(X) = AX - XA$.

(a) [2 points] Show that if A and B are diagonal, then $T_A T_B = T_B T_A$.

(b) [4 points] Give a basis $\boldsymbol{S}$ for V such that $[T_A]_{\boldsymbol{S}}$ is diagonal, for each diagonal matrix $A \in V$.

Consider the diagonal matrix A_o, with entries $a_1, \ldots, a_n$ such that $a_i = 1$ for $i = 1, \ldots, n-1$ and $a_n = 0$.

(c) [2 points] Use (b) to find the minimal polynomial of T_{A_o}.

(d) [2 points] Use (b) to find the characteristic polynomial of T_{A_o}.

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(2) (a) [3 points] Let W be a vector space over a field F and let T be a linear operator on W with minimal polynomial of the form f^r, where f is irreducible (monic) over F and r is a positive integer ≥ 1. Prove that $\exists\, \alpha \in W$ such that the T-annihilator of α is equal to f^r (i.e., $p_\alpha = f^r$).

(b) Let V be a finite-dimensional vector space over a field F and let T be a linear operator on V with minimal polynomial $p = f_1^{r_1} \ldots f_k^{r_k}$ where the f_i are distinct irreducible monic polynomials over F and the r_i are positive integers ≥ 1.

(b-1) [2 points] Announce the primary decomposition theorem.

(b-2) [5 points] Use (a) and the primary decomposition theorem to prove that $\exists\, \alpha \in V$ such that the T-annihilator of α is equal to p (i.e., $p_\alpha = p$).

(3) Let a , b, and c be elements of $\mathbb{R}$ and let A be the following matrix (over $\mathbb{R}$)

$$\begin{pmatrix} 0 & 0 & c \\ 1 & 0 & b \\ 0 & 1 & a \end{pmatrix}.$$

Let T be the linear operator associated to A in the standard basis e_1, e_2, e_3 of $\mathbb{R}^3$.

(a) [3 points] Does T have a cyclic vector?

(b) [3 points] Use (a) to find the minimal polynomial $\boldsymbol{p}$ of T.

(c) [2 points] Give numerical values for a, b, and c, where T will be triangulable but *not* diagonalizable.

(d) [2 points] Give numerical values for a, b, and c, where T will be diagonalizable.

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(4) Let V be an n-dimensional vector space over a field F.

(a) [5 points] Let N be a *nilpotent* linear operator on V with $N^{n-1} \neq 0$. Prove that N has a cyclic vector α and give the matrix of N in the standard basis of $Z(\alpha, N)$.

(b) [5 points] Let T be a linear operator on V which has n distinct characteristic values $c_1, \ldots, c_n$ with characteristic vectors $\alpha_1, \ldots, \alpha_n$, respectively. Prove that $\alpha = \alpha_1 + \ldots + \alpha_n$ is a cyclic vector for T.

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(5) Give all possible 7×7 complex matrices in rational form with minimal polynomial $(x - 1)^2(x - 2)$ and with three invariant factors.

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(6) Give the Jordan form, for a linear operator T, with minimal polynomial $x^2(x - 1)^2$ and characteristic polynomial $x^4(x - 1)^4$, such that nullity(T) = 2 and nullity($T - I$) = 3.