King Fahd University of Petroleum & Minerals
Department of Mathematics and Statistics

Math 550 - Linear Algebra (Term 181)

Exam 1

Notes:

- Duration = **3 hours.**
- Each problem is worth **10 points.**

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(1) In the following, determine whether the linear operator T is diagonalizable. If affirmative, find a basis with respect to which T has a diagonal matrix.

(a) [3 points] $T(x,y) = (-y,x),\ \forall\ (x,y) \in \mathbb{C}^2$.

(b) [3 points] $T(x,y) = (y,0),\ \forall\ (x,y) \in \mathbb{C}^2$.

(c) [4 points] $T(x,y,z) = (2x+y, 5y+3z, 8z),\ \forall\ (x,y,z) \in \mathbb{R}^3$.

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(2) Let V denote the space of polynomial functions over $\mathbb{R}$ and let $End(V)$ denote the space of linear operators on V.

(a) [2 points] Give explicit examples of T_1 and $T_2 \in End(V)$ with $T_1T_2 \neq T_2T_1$.

(b) [2 points] Give an example of $T \in End(V)$ which is non-singular and non-invertible.

(c) [6 points] Let $p \in V$ with $\text{degree}(p) = 2$ and let $T \in End(V)$ defined by $Tf = (pf)''$. Prove that T is an isomorphism. [**For the proof of the surjection, the use of linear algebra is compulsory**]

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(3) Let V be a finite-dimensional vector space and let $T \in End(V)$. Consider the following assertions:

(i) $rank(T^2) = rank(T)$;
(ii) $range(T) \cap nullspace(T) = 0$;
(iii) $nullspace(T^2) \subseteq nullspace(T)$.

(a) [6 points] Prove $(i) \Longrightarrow (ii)$.

(b) [2 points] Prove $(ii) \Longrightarrow (iii)$.

(c) [2 points] Prove $(iii) \Longrightarrow (i)$.

(4) Let T be a linear operator on $\mathbb{R}^3$.

(a) [5 points] Prove that if $T^3 = 0$ and $T^2 \neq 0$, then there is an ordered basis S for V such that $[T]_S = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

(b) [5 points] Prove that if A and B are two 3×3 matrices over $\mathbb{R}$ such that $A^2 \neq 0$ and $B^2 \neq 0$ with $A^3 = B^3 = 0$, then A and B are necessarily similar.

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(5) Let T be the linear operator on $\mathbb{R}^4$ which is represented in the standard ordered basis by the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & d \end{pmatrix}.$$

Under what conditions on a, b, c and d is T diagonalizable?

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(6) Let f and p denote the *characteristic* and *minimal* polynomials, respectively, of a given matrix. Find f and p for the following matrices.

(a) [2 points] $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ over $\mathbb{R}$.

(b) [2 points] $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ over $\mathbb{R}$.

(c) [6 points] $A = \begin{pmatrix} 0 & 0 & 0 & 0 & -3 \\ 1 & 0 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$ over $\mathbb{C}$.