Q:1 (20 points) Use separation of variables method to solve the problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \ 0 < x < \pi, \ t > 0,$$

subject to the boundary and initial conditions

$$u(0,t) = 0, \ u(\pi,t) = 0, \ t > 0$$
$$u(x,0) = \frac{x}{2}, \ 0 < x < \pi.$$

Sol.: Let $u(x,t) = X(x)T(t) \Rightarrow X''T = XT' \Rightarrow \frac{X''}{X} = \frac{T'}{T} = -\lambda$

$u(0,t) = 0 \Rightarrow X(0)T(t) = 0 \Rightarrow X(0) = 0.$

$u(\pi,t) = 0 \Rightarrow X(\pi)T(t) = 0 \Rightarrow X(\pi) = 0.$

(i) $\frac{T'}{T} = -\lambda \Rightarrow T' + \lambda T = 0 \Rightarrow T(t) = c\,e^{-\lambda t}.$

(ii) $\frac{X''}{X} = -\lambda \Rightarrow X'' + \lambda X = 0$, $X(0) = 0$ and $X(\pi) = 0$

(a) $\lambda = 0$: $X'' = 0 \Rightarrow X(x) = c_1 + c_2 x$

$X(0) = 0 \Rightarrow c_1 = 0 \Rightarrow X(x) = c_2 x$

$X(\pi) = 0 \Rightarrow c_2\pi = 0 \Rightarrow c_2 = 0 \Rightarrow X(x) = 0$, trivial solution

(b) $\lambda = -\alpha^2$ $(\alpha \neq 0)$; $X'' - \alpha^2 X = 0 \Rightarrow X = c_1\cosh\alpha x + c_2\sinh\alpha x$

$X(0) = 0 \Rightarrow c_1 = 0 \Rightarrow X(x) = c_2\sinh\alpha x$

$X(\pi) = 0 \Rightarrow c_2\sinh\alpha\pi = 0 \Rightarrow c_2 = 0 \Rightarrow X(x) = 0$ trivial solution

(c) $\lambda = \alpha^2$ $(\alpha \neq 0)$: $X'' + \alpha^2 X = 0 \Rightarrow X(x) = c_1\cos\alpha x + c_2\sin\alpha x$

$X(0) = 0 \Rightarrow c_1 = 0 \Rightarrow X(x) = c_2\sin\alpha x$

$X(\pi) = 0 \Rightarrow c_2\sin\alpha\pi = 0 \Rightarrow$ We take $c_2 \neq 0$ and $\sin\alpha\pi = 0$

$\Rightarrow \alpha\pi = n\pi \Rightarrow \alpha = n$, $n = 1, 2, 3, \ldots$

$X_n(x) = c_2\sin nx$ and $T_n(t) = c\,e^{-n^2 t}$

Product solutions are

$u_n(x,t) = (c_2\sin nx)(c\,e^{-n^2 t}) = A_n e^{-n^2 t}\sin nx$, $n = 1, 2, 3, \ldots$

By superposition principle, the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-n^2 t}\sin nx$$

Using $u(x,0) = f(x) = \frac{x}{2} \Rightarrow \frac{x}{2} = \sum_{n=1}^{\infty} A_n\sin nx$ (sine series)

$$A_n = \frac{2}{\pi}\int_0^{\pi}\frac{x}{2}\sin nx\,dx = \frac{1}{\pi}\left[-\frac{x}{n}\cos nx\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\cos n\pi + 0 + 0\right] = -\frac{1}{n}(-1)^n$$

Hence

$$u(x,t) = \sum_{n=1}^{\infty}\left(-\frac{1}{n}\right)(-1)^n e^{-n^2 t}\sin nx.$$

Q:2 (22 points) Use separation of variables method to solve the problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \ 0 < x < a, \ 0 < y < b,$$

subject to the boundary and initial conditions

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \ \left.\frac{\partial u}{\partial x}\right|_{x=a} = 0$$

$$u(x,0) = 0, \ u(x,b) = x.$$

Sol:- Let $u(x,y) = X(x)Y(y)$. Then $\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$

$\Rightarrow \quad X'' + \lambda X = 0$ and $Y'' - \lambda Y = 0$

$u_x|_{x=0} = 0 \Rightarrow X'(0) = 0$ $\qquad u(x,0) = 0 \Rightarrow Y(0) = 0.$

$u_x|_{x=a} = 0 \Rightarrow X'(a) = 0.$

(i) $\lambda = 0$: $X = c_1 + c_2 x$

$X'(0) = 0 \Rightarrow c_2 = 0.$ $\qquad X(x) = c_1$ is a solution for $\lambda = 0$.

(ii) $\lambda = -\alpha^2 (\alpha \neq 0)$: $X'' - \alpha^2 X = 0 \Rightarrow X = c_3 \cosh\alpha x + c_4 \sinh\alpha x$

$X' = \alpha c_3 \sinh\alpha x + \alpha c_4 \cosh\alpha x$

$X'(0) = 0 \Rightarrow c_4 = 0$

$X'(0) = 0 \Rightarrow c_3 = 0.$ $\qquad X(x) = 0$ trivial solution.

(iii) $\lambda = \alpha^2 \ (\alpha \neq 0)$; $X = c_5 \cos\alpha x + c_6 \sin\alpha x$

$X' = -\alpha c_5 \sin\alpha x + \alpha c_6 \cos\alpha x$

$X'(0) = 0 \Rightarrow c_6 = 0$; $X'(a) = 0 \Rightarrow \sin\alpha a = 0$

$\Rightarrow \alpha a = n\pi, \ n = 1, 2, 3, \ldots$

$\Rightarrow \alpha = \frac{n\pi}{a}$

Solutions are $X(x) = c_1$ and $X_n(x) = c_5 \cos\frac{n\pi}{a}x$, $n = 1, 2, 3, \ldots$

Now: $Y'' - \lambda Y = 0$

(a) $\lambda = 0 \Rightarrow Y'' = 0 \Rightarrow Y = c_2 + c_3 y$

$Y(0) = 0 \Rightarrow c_2 = 0.$ $\Rightarrow Y = c_3 y$

(b) $\lambda = \frac{n^2\pi^2}{a^2} \Rightarrow Y'' - \frac{n^2\pi^2}{a^2} Y = 0$

$\Rightarrow Y = c_4 \cosh\frac{n\pi}{a}y + c_6 \sinh\frac{n\pi}{a}y$

$Y(0) = 0 \Rightarrow c_4 = 0$ $\qquad \Rightarrow Y = c_6 \sinh\frac{n\pi}{a}y$

Product solutions are

$A_0 y$, $A_n \sinh\frac{n\pi}{a}y \cos\frac{n\pi}{a}x$, where $A_0 = c_1 c_3$, $A_n = c_5 c_6$

The superposition principle yields

$$u(x,y) = A_0 y + \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{a}y\right) \cos\left(\frac{n\pi}{a}x\right).$$

Using $u(x,b) = x$, we get

$$x = A_0 b + \sum_{n=1}^{\infty} A_n \sinh\frac{n\pi b}{a} \cos\frac{n\pi}{a}x. \qquad (\text{Cosine series})$$

$$\Rightarrow 2A_0 b = \frac{2}{a}\int_0^a x\,dx = \frac{2}{a}\cdot\frac{a^2}{2} = a$$

$$\Rightarrow A_0 = \frac{a}{2b}$$

and

$$A_n \sinh\frac{n\pi b}{a} = \frac{2}{a}\int_0^a x\cos\frac{n\pi}{a}x\,dx$$

$$= \frac{2}{a}\left[x\cdot\frac{a}{n\pi}\sin\frac{n\pi}{a}x\Big|_0^a - \int_0^a \frac{a}{n\pi}\sin\frac{n\pi}{a}x\,dx\right]$$

$$= \frac{2}{n\pi}\cdot\frac{a}{n\pi}\left[\cos\frac{n\pi}{a}x\right]_0^a$$

$$= \frac{2a}{n^2\pi^2}\left[(-1)^n - 1\right]$$

Hence

$$u(x,y) = \frac{ay}{2b} + \sum_{n=1}^{\infty} \frac{2a}{n^2\pi^2}\left[(-1)^n - 1\right] \sinh\frac{n\pi y}{a} \cos\frac{n\pi}{a}x \frac{1}{\sinh\frac{n\pi b}{a}}$$

Q:3 (16 points) Expand $f(x) = 1.\ \ 0 < x < 4$ in a Fourier-Bessel series using Bessel functions of order zero that satisfy the boundary condition $3J_0(4\alpha) + 4\alpha J_0'(4\alpha) = 0$.

<u>Solution</u>: $$c_i = \frac{2\alpha_i^2}{(\alpha_i^2 b^2 - n^2 + h^2) J_n^2(\alpha_i b)} \int_0^b x J_n(\alpha_i x) f(x) dx \; ; \; h J_n(\alpha b) + \alpha b J_n'(\alpha b) = 0$$

Here $n = 0,\ h = 3,\ b = 4$

$$c_i = \frac{2\alpha_i^2}{(16\alpha_i^2 + 9) J_0^2(4\alpha_i)} \int_0^4 x J_0(\alpha_i x) \cdot 1\, dx$$

Let $t = \alpha_i x$. Then $dx = \frac{dt}{\alpha_i}$

$$c_i = \frac{2\alpha_i^2}{(16\alpha_i^2 + 9) J_0^2(4\alpha_i)} \int_0^{4\alpha_i} \frac{t}{\alpha_i} J_0(t) \cdot \frac{dt}{\alpha_i}$$

$$= \frac{2}{(16\alpha_i^2 + 9) J_0^2(4\alpha_i)} \int_0^{4\alpha_i} \frac{d}{dt}[t J_1(t)]\, dt$$

(by recurrence relation)

$$= \frac{2}{(16\alpha_i^2 + 9) J_0^2(4\alpha_i)} \left[t J_1(t)\right]_0^{4\alpha_i}$$

$$= \frac{8\alpha_i J_1(4\alpha_i)}{(16\alpha_i^2 + 9) J_0^2(4\alpha_i)}$$

Thus
$$f(x) = \sum_{i=1}^{\infty} \frac{8\alpha_i J_1(4\alpha_i)}{(16\alpha_i^2 + 9) J_0^2(4\alpha_i)} J_0(\alpha_i x)$$

Q:4 (20 points) Use separation of variables to solve the problem

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=0,\ 0<r<3,\ 0<z<4,$$

subject to the boundary conditions

$$u(3,z)=0,\ 0<z<4,$$

$$u(r,0)=0,\ 0<r<3,$$

$$u(r,4)=2,\ 0<r<3.$$

Solution $u(r,z)$ is bounded at $r=0$.

Sol: Let $u(r,z)=R(r)Z(z)$. We find $R''Z+\frac{1}{r}R'Z+RZ''=0$.

$$\Leftrightarrow \frac{R''+\frac{1}{r}R'}{R}=-\frac{Z''}{Z}=-\lambda \Leftrightarrow R''+\frac{1}{r}R'+\lambda R=0,\ Z''-\lambda Z=0.$$

$u(3,z)=0 \Rightarrow R(3)=0$

$u(r,0)=0 \Rightarrow Z(0)=0$. Thus

$$\boxed{\begin{array}{l} rR''+R'+\lambda r R=0;\ R(3)=0 \\ Z''-\lambda Z=0;\ Z(0)=0. \end{array}}$$

Let $\lambda=\alpha^2$. Then $rR''+R'+\alpha^2 rR=0$

$\Rightarrow R(r)=c\,J_0(\alpha r)$.

and $R(3)=0 \Rightarrow J_0(3\alpha)=0$. $J_0(3\alpha_i)=0 \Rightarrow R=J_0(\alpha_i r)$.

α_i's are the non-zero values such that

$\lambda=\alpha_i^2$: $Z''-\alpha_i^2 Z=0 \Rightarrow c_1\cosh\alpha_i z+c_2\sinh\alpha_i z=Z$

$Z(0)=0 \Rightarrow c_1=0$ and $Z=c_2\sinh\alpha_i z$

Thus $u(r,z)=\sum_{i=1}^{\infty} A_i \sinh\alpha_i z\, J_0(\alpha_i r)$

$$u(r,4)=2 \Rightarrow 2=\sum_{i=1}^{\infty} A_i\sinh 4\alpha_i\, J_0(\alpha_i r)$$

$$\Rightarrow A_i\sinh 4\alpha_i=\frac{2}{9J_1^2(3\alpha_i)}\int_0^3 2r J_0(\alpha_i r)\,dr$$

$$=\frac{2}{9J_1^2(3\alpha_i)}\cdot\frac{2}{\alpha_i^2}\int_0^{3\alpha_i} t\,J_0(t)\,dt$$

$$=\frac{4}{9\alpha_i^2 J_1^2(3\alpha_i)}\Big[t\,J_1(t)\Big]_0^{3\alpha_i}$$

$$=\frac{4\cdot 3\alpha_i}{9\alpha_i^2 J_1^2(3\alpha_i)}J_1(3\alpha_i)$$

$$A_i=\frac{4}{3\sinh 4\alpha_i\, J_1(3\alpha_i)\cdot\alpha_i}$$

Hence

$$u(r,z)=\sum_{i=1}^{\infty}\frac{4\sinh\alpha_i z}{3\alpha_i\sinh 4\alpha_i\, J_1(3\alpha_i)}\, J_0(\alpha_i r).$$

Q:5 (20 points) Use Laplace transform to solve the problem

$$\frac{\partial^2 u}{\partial x^2} = 3 + \frac{\partial^2 u}{\partial t^2}, \ x > 0, \ t > 0,$$

subject to the boundary and initial conditions

$$u(0,t) = 0, \ \lim_{x\to\infty} \frac{\partial u}{\partial x} = 0, \ t > 0,$$

$$u(x,0) = 0, \ \left.\frac{\partial u}{\partial t}\right|_{t=0} = e^{-x}, \ x > 0.$$

Sol:

$$\frac{d^2 U}{dx^2} = \frac{3}{s} + s^2 U - s\,u(x,0) - u_t(x,0)$$

$$\frac{d^2 U}{dx^2} - s^2 U = \frac{3}{s} - e^{-x}$$

$$\mathcal{L}\{u(0,t)\} = U(0,s) = 0 \ ; \ \mathcal{L}\left\{\lim_{x\to\infty} \frac{\partial u}{\partial x}\right\} = \lim_{x\to\infty} \frac{dU}{dx} = 0 .$$

$U_c(x,s) = c_1 e^{-sx} + c_2 e^{sx}$. Let $U_p = A + Be^{-x}$ Then $A = -\frac{3}{s^3}$

$B = \frac{1}{s^2-1}$

$$U = U_c + U_p$$

$$= c_1 e^{-sx} + c_2 e^{sx} - \frac{3}{s^3} + \frac{1}{s^2-1} e^{-x}$$

$\lim_{x\to\infty} \frac{dU}{dx} = 0 \Rightarrow c_2 = 0.$

$U(0,s) = 0 \Rightarrow c_1 = \frac{3}{s^3} - \frac{1}{s^2-1}$

Therefore, $U(x,s) = \frac{3}{s^3} e^{-sx} - \frac{1}{s^2-1} e^{-xs} - \frac{3}{s^3} - \frac{1}{s^2-1} e^{-x}$

Inverting

$$u(x,t) = \frac{3}{2}(t-x)^2 \mathcal{U}(t-x) - \sinh(t-x)\,\mathcal{U}(t-x) - \frac{3}{2}t^2 - \sinh t \, e^{-x}$$

Q:6 (20 points) Solve the problem using the Fourier **cosine** transform

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \ 0 < x < 1, \ y > 0,$$

$$u(0,y) = 0, \ y > 0, \quad u(1,y) = \begin{cases} 1, & 0 < y < 1 \\ 0, & y > \text{[redacted]}, \end{cases}$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0, \ 0 < x < 1.$$

Sol: Using Fourier cosine transform with respect to the variable y

$$\mathcal{F}_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} + \mathcal{F}_c\left\{\frac{\partial^2 u}{\partial y^2}\right\} = 0$$

$$\frac{d^2 U}{dx^2} + \left(-\alpha^2 U(x,\alpha) - u_y(x,0)\right) = 0$$

$$\frac{d^2 U}{dx^2} - \alpha^2 U = 0$$

$$U(x,\alpha) = c_1 \cosh \alpha x + c_2 \sinh \alpha x$$

$$U(0,\alpha) = 0 \Rightarrow c_1 = 0 \qquad \text{and } U(x,\alpha) = c_2 \sinh \alpha x$$

Now, $\mathcal{F}_c\{u(1,y)\} = U(1,\alpha) = \int_0^\infty f(y) \cos \alpha y \, dy$

$$= \int_0^1 1 \cdot \cos \alpha y \, dy$$

$$= \left.\frac{\sin \alpha y}{\alpha}\right|_0^1 = \frac{\sin \alpha}{\alpha}$$

[redacted] $U(1,\alpha) = c_2 \sinh \alpha = \frac{\sin \alpha}{\alpha} \Rightarrow c_2 = \frac{\sin \alpha}{\alpha \sinh \alpha}$

Hence $U(x,\alpha) = \frac{\sin \alpha}{\alpha \sinh \alpha} \cdot \sinh \alpha x$

$$\Rightarrow u(x,y) = \frac{2}{\pi} \int_0^\infty \frac{\sin \alpha}{\alpha \sinh \alpha} \sinh \alpha x \cos \alpha y \, d\alpha.$$

Q:7 (22 points) Find the steady-state temperature $u(r,\theta)$ in a sphere of unit radius by solving the problem

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}=0,\ 0<r<1,\ 0<\theta<\pi,$$

$$u(1,\theta)=2-\frac{4}{3}\cos(\theta).$$

Let $u=R(r)\,\Theta(\theta)$. Then $R''\Theta+\frac{2}{r}R'\Theta+\frac{1}{r^2}R\Theta''+\frac{\cot\theta}{r^2}R\Theta'=0.$

$$\frac{R''+\frac{2}{r}R'}{R}=-\frac{1}{r^2}\,\frac{\Theta''+\cot\theta\,\Theta'}{\Theta}$$

$$\frac{r^2R''+2rR'}{R}=-\frac{\Theta''+\cot\theta\,\Theta'}{\Theta}=\lambda$$

$\Rightarrow \Theta''+\cot\theta\,\Theta'+\lambda\Theta=0$ and $r^2R''+2rR'-\lambda R=0$.

Let $x=\cos\theta$. Then

$(1-x^2)\,\Theta''(x)-2x\,\Theta'(x)+\lambda\,\Theta(x)=0$; $\lambda=n(n+1)$, $n=0,1,2,\ldots$

$\Theta_n(x)=P_n(x)$, $\Theta_n(\theta)=P_n(\cos\theta)$.

$r^2R''+2rR'-\lambda R=0$, $\lambda_n=n(n+1)$, Cauchy-Euler equation.

$m^2+m-n(n+1)=0$; $m=n,\ -(n+1)$

$R=c_1r^n+c_2r^{-(n+1)}$; $u(r,\theta)$ is bounded $\Rightarrow c_2=0$.

$$u(r,\theta)=\sum_{n=0}^{\infty}A_n r^n P_n(\cos\theta)$$

$u(1,\theta)=2-\frac{4}{3}\cos\theta \Rightarrow 2-\frac{4}{3}\cos\theta=\sum_{n=0}^{\infty}A_nP_n(\cos\theta)$

$$\Rightarrow A_n=\frac{2n+1}{2}\int_0^{\pi}\left(2-\frac{4}{3}\cos\theta\right)\cdot P_n(\cos\theta)\sin\theta\,d\theta.$$

$$=\frac{2n+1}{2}\int_0^{\pi}\underset{2P_0(\cos\theta)}{\underset{\shortparallel}{2}}\,P_n(\cos\theta)\sin\theta\,d\theta-\frac{2n+1}{2}\int_0^{\pi}\frac{4}{3}\underset{P_1(\cos\theta)}{\underset{\Downarrow}{\cos\theta}}\,P_n(\cos\theta)\sin\theta\,d\theta$$

By orthogonality $A_n=0$ for $n\neq 0,1$

$A_0=\frac{1}{2}\cdot 2\int_0^{\pi}P_0(\cos\theta)P_0(\cos\theta)\sin\theta\,d\theta$

$=\int_{-1}^{1}P_0(x)P_0(x)\,dx=\int_{-1}^{1}dx=2$

$A_1=-\frac{3}{2}\cdot\frac{4}{3}\int_0^{\pi}P_1(\cos\theta)P_1(\cos\theta)\sin\theta\,d\theta$

$=-2\int_0^{\pi}\cos^2\theta\sin\theta=2\left[\frac{\cos^3\theta}{3}\right]_0^{\pi}$

$=2\left[-\frac{1}{3}-\frac{1}{3}\right]=-\frac{4}{3}$

$u(r,\theta)=A_0P_0(\cos\theta)+A_1rP_1(\cos\theta)$

$=2-\frac{4}{3}r\cos\theta$