Q:1 (6 + 6 = 12 points) Evaluate

(a) $\mathcal{L}\{e^{2t}\ t^2\ \sin(t)\}$ (b) $\mathcal{L}^{-1}\{\frac{(2s+1)e^{-2s}}{s^2+4s+5}\}$.

(a) $\mathcal{L}\{\sin t\} = \dfrac{1}{s^2+1}$, $\mathcal{L}\{t^2 \sin t\} = \dfrac{d^2}{ds^2}\left\{\dfrac{1}{s^2+1}\right\}$

$$\frac{d}{ds}\left\{\frac{1}{s^2+1}\right\} = \frac{-2s}{(s^2+1)^2}$$

$$\frac{d^2}{ds^2}\left\{\frac{1}{s^2+1}\right\} = -\frac{d}{ds}\left\{\frac{2s}{(s^2+1)^2}\right\}$$

$$= -\left[\frac{2(s^2+1)^2 - 2s \cdot 2(s^2+1) \cdot 2s}{(s^2+1)^4}\right] = \frac{6s^2-2}{(s^2+1)^3}$$

$$\mathcal{L}\{e^{2t}\ t^2 \sin t\} = \frac{6(s-2)^2 - 2}{((s-2)^2+1)^3}$$

(b) $\mathcal{L}^{-1}\left\{\dfrac{2s+1}{s^2+4s+5}\right\} = \mathcal{L}^{-1}\left\{\dfrac{2s+1}{(s+2)^2+1}\right\}$

$$= 2\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{3}{(s+2)^2+1}\right\}$$

$$= 2e^{-2t}\cos t - 3e^{-2t}\sin t$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{(2s+1)e^{-2s}}{s^2+4s+5}\right\} = \left[2e^{-2(t-2)}\cos(t-2) - 3e^{-2(t-2)}\sin(t-2)\right] \times u(t-2)$$

Q:2 (10 points) Use **Convolution Theorem** to find inverse Laplace transform of

$$G(s) = \frac{s}{(s^2+1)^2}.$$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= \cos t * \sin t$$

$$= \int_0^t \cos\tau \sin(t-\tau)\,d\tau$$

$$= \frac{1}{2}\int_0^t \left[\sin t - \sin(2\tau - t)\right] d\tau \qquad *$$

$$= \frac{1}{2}\left[\tau \sin t + \frac{1}{2}\cos(2\tau - t)\right]_0^t$$

$$= \frac{1}{2}\left[t \sin t + \frac{1}{2}\cos t - 0 - \frac{1}{2}\cos(-t)\right]$$

$$= \frac{1}{2} t \sin t$$

$*$ or $\quad \frac{1}{2}\int_0^t \left[\sin t + \sin(t - 2\tau)\right] d\tau$

Q:3 (12 points) Solve $y' + y = 1 + \int_0^t e^{(t-\tau)} y(\tau) d\tau, \quad y(0) = 0.$

Sol: Laplace transform on both sides, we get

$$sY(s) - y(0) + Y(s) = \frac{1}{s} + \mathcal{L}\{e^t * y(t)\}$$

$$sY(s) - 0 + Y(s) = \frac{1}{s} + \frac{1}{s-1} Y(s)$$

$$(s+1) Y(s) = \frac{1}{s} + \frac{1}{s-1} Y(s)$$

$$\left(s+1 - \frac{1}{s-1}\right) Y(s) = \frac{1}{s}$$

$$\frac{(s^2-2)}{s-1} Y(s) = \frac{1}{s}$$

$$Y(s) = \frac{s-1}{s(s^2-2)}$$

$$= \frac{1}{s^2-2} - \frac{1}{s(s^2-2)}$$

Inverting

$$y(t) = \frac{1}{\sqrt{2}} \sinh \sqrt{2}\, t - 1 * \frac{1}{\sqrt{2}} \sinh \sqrt{2}\, t$$

$$= \frac{1}{\sqrt{2}} \sinh \sqrt{2}\, t - \frac{1}{\sqrt{2}} \int_0^t \sinh \sqrt{2}\, t \, dt$$

$$= \frac{1}{\sqrt{2}} \sinh \sqrt{2}\, t - \left.\frac{\cosh \sqrt{2}\, t}{2}\right|_0^t$$

$$= \frac{1}{\sqrt{2}} \sinh \sqrt{2}\, t - \frac{1}{2} \cosh \sqrt{2}\, t + \frac{1}{2}$$

OR $\cancel{\mathcal{L}^{-1}\left\{\frac{1}{s(s^2-2)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s+\sqrt{2})(s-\sqrt{2})}\right\}}$

$\cancel{= \mathcal{L}^{-1}\left\{\frac{1}{2s}\right\}}$

Q:4 (12 points) Solve the IVP : $y'' + y' + 2y = \delta(t + 7)$, $y(0) = 0$, $y'(0) = 0$.

Sol: Taking L.T. on both sides, we get

$$s^2 Y(s) - s\,y(0) - y'(0) + sY(s) - y(0) + 2Y(s) = e^{7s}$$

$$(s^2 + s + 2)\,Y(s) = e^{7s}$$

$$Y(s) = \frac{e^{7s}}{s^2+s+2}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{e^{7s}}{s^2+s+2}\right\}$$

Now $\mathcal{L}^{-1}\left\{\frac{1}{s^2+s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+\frac{1}{2})^2+\frac{7}{4}}\right\}$

$$= \frac{2}{\sqrt{7}}\,\mathcal{L}^{-1}\left\{\frac{\frac{\sqrt{7}}{2}}{(s+\frac{1}{2})^2+7/4}\right\}$$

$$= \frac{2}{\sqrt{7}}\,e^{-\frac{t}{2}}\sin\frac{\sqrt{7}}{2}t$$

Therefore,

$$y(t) = \mathcal{L}^{-1}\left\{\frac{e^{7s}}{s^2+s+2}\right\}$$

$$= \frac{2}{\sqrt{7}}\,e^{-\frac{1}{2}(t+7)}\sin\left(\frac{\sqrt{7}}{2}(t+7)\right)u(t+7)$$

Q:5 (14 points) Show that the set of functions $\{\frac{1}{2}, \cos(n\pi x)\}, n = 1, 2, 3, \cdots$ is orthogonal on $[0, 4]$. Also, find the **norm** of each function.

Sol:
$$(f_1, f_2) = \int_0^4 \frac{1}{2} \cos n\pi x \, dx = \frac{\sin n\pi x}{2n\pi}\Big|_0^4 = 0.$$

$$(f_n, f_m) = \int_0^4 \cos n\pi x \cos m\pi x \, dx \qquad (n \neq m)$$

$$= \frac{1}{2}\int_0^4 \left[\cos(n+m)\pi x + \cos(n-m)\pi x\right] dx$$

$$= \frac{1}{2}\left[\frac{\sin(n+m)\pi x}{(n+m)\pi} + \frac{\sin(n-m)\pi x}{(n-m)\pi}\right]_0^4 = 0$$

$$\|f_1\|^2 = \int_0^4 \frac{1}{2}\cdot\frac{1}{2}\, dx = \frac{1}{4}x\Big|_0^4 = 1$$

$$\|f_n\|^2 = \int_0^4 \cos^2 n\pi x \, dx$$

$$= \frac{1}{2}\int_0^4 \left[1 + \cos 2n\pi x\right] dx$$

$$= \frac{1}{2}\left[x + \frac{\sin 2n\pi x}{2n\pi}\right]_0^4$$

$$= \frac{1}{2}\cdot 4 = 2.$$

$$\|f_n\| = \sqrt{2}$$

10+6=

Q:6 (16 points) (a) Find the Fourier series of the function $f(x) = \begin{cases} 1, & -1 < x < 0 \\ x, & 0 \le x < 1. \end{cases}$

(b) Use the result of the Part(a) to calculate the value of the series:

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

Sol:

$$a_0 = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} 1\,dx + \int_{0}^{1} x\,dx = x\Big|_{-1}^{0} + \frac{x^2}{2}\Big]_0^1 = 1 + \frac{1}{2} = \frac{3}{2}$$

$$a_n = \int_{-1}^{0} \cos n\pi x\,dx + \int_{0}^{1} x\cos n\pi x\,dx$$

$$= \frac{\sin n\pi x}{n\pi}\Big|_{-1}^{0} + \left[\frac{x\sin n\pi x}{n\pi}\Big|_0^1 - \int_0^1 \frac{\sin n\pi x}{n\pi}\,dx\right]$$

$$= 0 + 0 + \frac{\cos n\pi x}{n^2\pi^2}\Big|_0^1 = \frac{\cos n\pi - 1}{n^2\pi^2}$$

$$= \frac{(-1)^n - 1}{n^2\pi^2}$$

$$b_n = \int_{-1}^{0} \sin n\pi x\,dx + \int_{0}^{1} x\sin n\pi x\,dx$$

$$= -\frac{\cos n\pi x}{n\pi}\Big|_{-1}^{0} + \left[-\frac{x\cos n\pi x}{n\pi}\Big|_0^1 + \int_0^1 \frac{\cos n\pi x}{n\pi}\,dx\right]$$

$$= \frac{1}{n\pi}\left[(-1)^n - 1\right] - \frac{1}{n\pi}[-1]^n + \frac{\sin n\pi x}{n^2\pi^2}\Big|_0^1$$

$$= -\frac{1}{n\pi}$$

Fourier series $f(x) = \frac{3}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2\pi^2}\cos n\pi x - \frac{1}{n\pi}\sin n\pi x\right]$

(b) At $x=0$, $f(x)$ is discontinuous and series will converge to

$$\frac{f(0^+) + f(0^-)}{2} = \frac{0+1}{2} = \frac{1}{2}$$

$$\frac{1}{2} = \frac{3}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2\pi^2}\cos n\pi\cdot 0 - \frac{1}{n\pi}\sin n\pi\cdot 0$$

$$= \frac{3}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2\pi^2}$$

$$-\frac{1}{4} = \frac{1}{\pi^2}\left[-\frac{2}{1^2} - \frac{2}{3^2} - \frac{2}{5^2} - \cdots\right]$$

or

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Q:7 (**10** points) Expand $f(x) = \cos(x), \quad 0 < x < \frac{\pi}{2}$ in a cosine series.

Sol:

$$a_0 = \frac{2}{\pi/2} \int_0^{\pi/2} \cos x \, dx$$

$$= \frac{4}{\pi} \Big[\sin x \Big]_0^{\pi/2} = \frac{4}{\pi}$$

$$a_n = \frac{4}{\pi} \int_0^{\pi/2} \cos x \cdot \cos \frac{n\pi}{\pi/2} x \, dx$$

$$= \frac{4}{\pi} \int_0^{\pi/2} \cos x \, \cos 2nx \, dx$$

$$= \frac{4}{\pi} \cdot \frac{1}{2} \int_0^{\pi/2} \Big[\cos (2n-1)x + \cos (2n+1)x \Big] dx$$

$$= \frac{2}{\pi} \Big[\frac{\sin (2n-1)x}{2n-1} + \frac{\sin (2n+1)x}{2n+1} \Big]_0^{\pi/2}$$

$$= \frac{2}{\pi} \Big[\frac{(-1)^{n+1}}{2n-1} - \frac{(-1)^{n+1}}{2n+1} \Big]$$

$$= \frac{2}{\pi} (-1)^{n+1} \Big[\frac{2n+1-2n+1}{4n^2-1} \Big]$$

$$= \frac{4}{\pi} \frac{(-1)^{n+1}}{4n^2-1}$$

$$\therefore \quad f(x) = \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2-1} \cos 2nx .$$

Q:8 (14 points) Find the eigenvalues and the eigenfunctions of the **Sturm-Liouville problem:**

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(\frac{\pi}{6}) = 0.$$

Case I: $\lambda = 0$; $y'' = 0$

$y = Ax + B$

$y(0) = 0 \Rightarrow B = 0$

$y(\frac{\pi}{6}) = 0 \Rightarrow A\frac{\pi}{6} = 0 \Rightarrow A = 0$

$\Rightarrow y = 0$ is a trivial solution.

$\Rightarrow \lambda = 0$ is not an eigenvalue.

Case II: $\lambda < 0$. Let $\lambda = -\alpha^2 < 0$ $(\alpha \neq 0)$, Then DE reduces to $y'' - \alpha^2 y = 0$

$y = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$

$y(0) = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$

$y(\frac{\pi}{6}) = 0 \Rightarrow c_1 e^{\frac{\alpha\pi}{6}} + c_2 e^{-\frac{\alpha\pi}{6}} = 0$

$c_1\left[e^{\frac{2\alpha\pi}{6}} - 1\right] = 0$

$c_1 \neq 0$, $e^{\frac{2\alpha\pi}{6}} - 1 = 0$ (Not possible).

$\Rightarrow c_1 = 0$ & $c_2 = 0$.

$\Rightarrow \lambda = -\alpha^2 < 0$ is not eigenvalue.

Case III: $\lambda = \alpha^2 > 0$ $(\alpha \neq 0)$; DE: $y'' + \alpha^2 y = 0$

$y = c_1 \cos\alpha x + c_2 \sin\alpha x$

$y(0) = 0 \Rightarrow c_1 = 0$

$y(\frac{\pi}{6}) = 0 \Rightarrow 0 = c_2 \sin\frac{\alpha\pi}{6}$

$\Rightarrow \sin\frac{\alpha\pi}{6} = 0$ $(c_2 \neq 0)$

$\Rightarrow \sin\frac{\alpha\pi}{6} = \sin n\pi$

$\Rightarrow \alpha = 6n$, $n = 1, 2, 3, \ldots$

Eigenvalues are $\lambda = 36n^2$, $n = 1, 2, 3, \ldots$

Eigenfunctions are $y_n = \sin 6nx$, $n = 1, 2, 3, \ldots$.