King Fahd University of Petroleum & Minerals
Department of Mathematics and Statistics

MATH 302, Semester 181 (2018-2019)

EXAM I

October 10, 2018

Allowed Time: 120 mins

Student Name:

Student ID Number:

Section Number:

Instructions:

1. Write neatly and legibly -- *you may lose points for messy work.*
2. Show all your work -- *no points for answers without justification.*
3. Programmable calculators and Mobiles are **not** allowed.
4. Make sure that you have 7 problems (7 pages + cover page + Scratch sheet).

Problem No.	Points	Out of
1		14
2		13
3		14
4		14
5		15
6		15
7		15
Total:		**100**

Problem 1. a. Check if the vectors $U = < 1,1,0 >$, $V = < 1,1,2 >$, and $W = < 2,2,1 >$ are linearly dependent or independent?

Sol. $\alpha U + \beta V + \gamma W = 0 \Leftrightarrow$

$< \alpha, \alpha, 0 > + < \beta, \beta, 2\beta > + < 2\gamma, 2\gamma, \gamma > = 0 \Leftrightarrow$

$< \alpha + \beta + 2\gamma, \alpha + \beta + 2\gamma, 2\beta + \gamma > = < 0, 0, 0 > \Leftrightarrow$

$\Rightarrow \begin{array}{l} \alpha + \beta + 2\gamma = 0 \\ 2\beta + \gamma = 0 \end{array} \Rightarrow 2\alpha + \cancel{2\beta} + 4\gamma - (\cancel{2\beta} + \gamma) = 0$

$\Rightarrow 2\alpha + 3\gamma = 0 \Rightarrow \alpha = -\dfrac{3\gamma}{2}$

Also, $2\beta + \gamma = 0 \Rightarrow \beta = -\dfrac{\gamma}{2}$

So, we can find $\gamma = 2$, $\beta = -1$, $\alpha = -3$ so that

$-3U - V + 2W = 0$

$\therefore$ U, V, W are linearly dependent.

b. Show that $S = \{X = < x, y, 0, z > \in \mathbb{R}^4 : xyz \geq 0\}$ is not a subspace of $\mathbb{R}^4$.

$U = < +1, +1, 0, 1 > \in S$ but $-U = < -1, -1, 0, -1 > \notin S$

Since $(-1)(-1)(-1) < 0$. Thus S is not a subspace.

Problem 2. According to the values of α, find the rank of the matrix

$$A = \begin{pmatrix} -1 & 2 & -3 \\ 1 & -2 & \alpha \\ 0 & -3 & 0 \\ -1 & -1 & -3 \end{pmatrix}$$

Sol.

$$\begin{pmatrix} -1 & 2 & -3 \\ 1 & -2 & \alpha \\ 0 & -3 & 0 \\ -1 & -1 & -3 \end{pmatrix} \xrightarrow[R_3 - R_1]{R_2 + R_1} \begin{pmatrix} -1 & 2 & -3 \\ 0 & 0 & \alpha-3 \\ 0 & -3 & 0 \\ 0 & -3 & 0 \end{pmatrix} \xrightarrow{R_4 - R_3}$$

$$\begin{pmatrix} -1 & 2 & -3 \\ 0 & 0 & \alpha-3 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow[\text{Not Necessary}]{R_2 \longleftrightarrow R_3} \begin{pmatrix} -1 & 2 & -3 \\ 0 & -3 & 0 \\ 0 & 0 & \alpha-3 \\ 0 & 0 & 0 \end{pmatrix}$$

So, if $\alpha = 3$ then rank $A = 2$

if $\alpha \neq 3$ then rank $A = 3$

Problem 3. Use the Gauss method to find all solutions of the system

$$\begin{cases} -3x + 2y - 6z = 6 \\ 5x + 7y - 5z = 6 \\ x + 4y - 2z = 8 \end{cases}$$

Sol.

$$\left(\begin{array}{ccc|c} -3 & 2 & -6 & 6 \\ 5 & 7 & -5 & 6 \\ 1 & 4 & -2 & 8 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 4 & -2 & 8 \\ 5 & 7 & -5 & 6 \\ -3 & 2 & -6 & 6 \end{array}\right) \xrightarrow[R_3+3R_1]{R_2-5R_1}$$

$$\left(\begin{array}{ccc|c} 1 & 4 & -2 & 8 \\ 0 & -13 & 5 & -34 \\ 0 & 14 & -12 & 30 \end{array}\right) \xrightarrow[R_3/2 \to R_3]{R_2+R_3 \to R_2} \left(\begin{array}{ccc|c} 1 & 4 & -2 & 8 \\ 0 & \boxed{1} & -7 & -4 \\ 0 & 7 & -6 & 15 \end{array}\right)$$

$$\xrightarrow[R_3-7R_2]{R_1-4R_2} \left(\begin{array}{ccc|c} 1 & 0 & 26 & 24 \\ 0 & 1 & -7 & -4 \\ 0 & 0 & 43 & 43 \end{array}\right) \xrightarrow{R_3/43} \left(\begin{array}{ccc|c} 1 & 0 & 26 & 24 \\ 0 & 1 & -7 & -4 \\ 0 & 0 & \boxed{1} & 1 \end{array}\right)$$

$$\xrightarrow[R_2+7R_3]{R_1-26R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right) \Rightarrow \text{the solution is}$$

$$X = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}.$$

Problem 4. Use elementary operations to find the inverse of

$$A=\begin{pmatrix}1&2&3\\2&5&3\\1&0&8\end{pmatrix}$$

<u>Sol.</u>

$$\left(\begin{array}{ccc|ccc}1&2&3&1&0&0\\2&5&3&0&1&0\\1&0&8&0&0&1\end{array}\right)\xrightarrow[R_3-R_1]{R_2-2R_1}\left(\begin{array}{ccc|ccc}1&2&3&1&0&0\\0&\boxed{1}&-3&-2&1&0\\0&-2&5&-1&0&1\end{array}\right)$$

$$\xrightarrow[R_3+2R_2]{R_1-2R_2}\left(\begin{array}{ccc|ccc}1&0&9&5&-2&0\\0&1&-3&-2&1&0\\0&0&-1&-5&2&1\end{array}\right).$$

$$\xrightarrow{-R_3}\left(\begin{array}{ccc|ccc}1&0&9&5&-2&0\\0&1&-3&-2&1&0\\0&0&1&5&-2&-1\end{array}\right)$$

$$\xrightarrow[R_2+3R_3]{R_1-9R_3}\left(\begin{array}{ccc|ccc}1&0&0&-40&16&9\\0&1&0&13&-5&-3\\0&0&1&5&-2&-1\end{array}\right)$$

So, $$A^{-1}=\begin{pmatrix}-40&16&9\\13&-5&-3\\5&-2&-1\end{pmatrix}$$

Problem 5.

a. Find all the eigenvalues of

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 5 & 6 \\ -3 & 0 & 7 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & 0 & 3 \\ 0 & 5-\lambda & 6 \\ -3 & 0 & 7-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 5-\lambda & 6 \\ 0 & 7-\lambda \end{vmatrix} + 3\begin{vmatrix} 0 & 5-\lambda \\ -3 & 0 \end{vmatrix}$$

$$= (1-\lambda)(5-\lambda)(7-\lambda) + 9(5-\lambda)$$

$$= (5-\lambda)(\lambda^2 - 8\lambda + 7 + 9)$$

$$= (5-\lambda)(\lambda-4)^2 = 0$$

$\Rightarrow \lambda = 4, 4, 5$ are the eigenvalues.

a. Is A diagonalizable and why?

We find eigenvector(s) of $\lambda = 4$ since it is repeated

$$\begin{pmatrix} -3 & 0 & 3 \\ 0 & 1 & 6 \\ -3 & 0 & 3 \end{pmatrix} \xrightarrow[R_3/-3]{R_1/-3} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 6 \\ 1 & 0 & -1 \end{pmatrix} \xrightarrow{R_3 - R_1} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow x = z$, $y = -6z$. So, an eigenvector is

$$K = \begin{pmatrix} 1 \\ -6 \\ 1 \end{pmatrix}$$

4 is repeated and we only got an eigenvector

$\Rightarrow$ A is not diagonalizable.

Problem 6. Given the matrix $A = \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}$. Find a nonsingular matrix P and a diagonal matrix D such that $P^{-1}AP = D$. Compute A^{21}.

Sol. $\begin{vmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 = (\lambda-1)(\lambda-2) = 0$

$\Rightarrow \lambda = 1, 2$ are the eigenvalues.

Eigenvectors

$\boxed{\lambda = 1}$ $\begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \xrightarrow{R_1/2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \xrightarrow{R_2+R_1} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow x = -y \Rightarrow K_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector

$\boxed{\lambda = 2}$ $\begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix} \xrightarrow{R_2+R_1} \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow x = -2y \Rightarrow K_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is an eigenvector

$P = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

We have $\cancel{D}\, A = PDP^{-1} \Rightarrow A^{21} = PD^{21}P^{-1}$

$A^{21} = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2^{21} \end{pmatrix} \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$

$= \begin{pmatrix} 1 & -2^{22} \\ -1 & 2^{21} \end{pmatrix} \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 1+2^{22} & -2+2^{22} \\ 1-2^{21} & 2-2^{21} \end{pmatrix}$

Problem 7.

a) Find all the values of k for which the matrix

$$A = k\begin{pmatrix} 1+\sqrt{2} & \sqrt{2} & \sqrt{2}-1 \\ -\sqrt{2} & 2 & \sqrt{2} \\ -1+\sqrt{2} & -\sqrt{2} & 1+\sqrt{2} \end{pmatrix}$$

is orthogonal

$\langle 1+\sqrt{2}, \sqrt{2}, \sqrt{2}-1\rangle \cdot \langle -\sqrt{2}, 2, \sqrt{2}\rangle = -\sqrt{2} - 2 + 2\sqrt{2} + 2 - \sqrt{2} = 0$

$\langle 1+\sqrt{2}, \sqrt{2}, \sqrt{2}-1\rangle \cdot \langle -1+\sqrt{2}, -\sqrt{2}, 1+\sqrt{2}\rangle = 1 - 2 + 1 = 0$

$\langle -\sqrt{2}, 2, \sqrt{2}\rangle \cdot \langle -1+\sqrt{2}, -\sqrt{2}, 1+\sqrt{2}\rangle = \sqrt{2} - 2 - 2\sqrt{2} + \sqrt{2} + 2 = 0$

The rows are mutually orthogonal.

We compute the magnitude of each row:

$\| k\langle 1+\sqrt{2}, \sqrt{2}, \sqrt{2}-1\rangle \|^2 = 8k^2$ (similarly for all)

$8k^2 = 1 \iff k = \pm \dfrac{1}{\sqrt{8}} = \pm \dfrac{\sqrt{2}}{4}$

For these values A is orthogonal.

b. Let

$M = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$, $\sigma = ab + bc + ac$, and $S = a + b + c$

Show that M is orthogonal if and only if $\sigma = 0$ and $S = \pm 1$.

$\langle a, b, c\rangle \cdot \langle c, a, b\rangle = ab + ba + cb = \sigma$

Similarly for $\langle a, b, c\rangle \cdot \langle b, c, a\rangle = \sigma$

and $\langle c, a, b\rangle \cdot \langle b, c, a\rangle = \sigma$

The magnitude of each column or row is $\sqrt{a^2+b^2+c^2}$

So M is orthogonal iff $\sigma = 0$ and $a^2+b^2+c^2 = 1$

$$S^2 = (a+b+c)^2 = a^2+b^2+c^2+2ab+2ac+2bc$$
$$= a^2+b^2+c^2+2\sigma = 1+0=1.$$
$$\Longrightarrow S = \pm 1.$$