KFUPM, DEPARTMENT OF MATHEMATICS AND STATISTICS

MATH 225: FINAL EXAM, SEMESTER (181), DECEMBER 22, 2018

Time: 07:00 to 10:00 pm

Name : ....

ID : ..... Section : .....

Exercise	**Points**
1	_____ 14
2	_____ 14
3	_____ 14
4	_____ 14
5	_____ 14
6	_____ 14
7	_____ 14
8	_____ 14
9	_____ 14
10	_____ 14
Total	_____ 140

Exercise 1 (14 pts)**.** Let

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}.$$

(1) Find the matrix $\exp(tA)$ (exponential of tA), for $t \in \mathbb{R}$.

(2) Solve the following initial value problem: $X'(t) = AX(t)$ with $X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, $x(0) = 1$ and $= y(0) = 2$.

Exercise 2 (14 pts)**.** Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

find a complex matrix B such that $B^2 = A$.

Exercise 3 (14 pts)**.**

(1) Find the distance between the point $P_0 = (1, 3)$ and the line L of equation $y = 2x$.

(2) Find the closest point Q of L to the point $P_0 = (1, 3)$.

Exercise 4 (14 pts)**.** Let $S = \{(x + y, x - y, x + 3y, x - 2y)^T : x, y \in \mathbb{R}\}$.

(1) Find an orthonormal basis of the subspace S of $\mathbb{R}^4$.

(2) Find the closest vector $\mathbf{p}$ of S to $\mathbf{w} = (1, 1, 1, 1)^T$.

Exercise 5 (14 pts)**.** Find a QR-decomposition of the matrix

$$A = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Exercise 6 (14 pts)**.** Let $\mathcal{B}_0 = (1, x, x^2, x^3)$ be the standard basis of $\mathbf{P}_3$.

(1) Show that $\mathcal{B}_1 = (1, 2+x, 2+x+x^2, 2+x+x^2+x^3)$ is a basis of $\mathbf{P}_3$.

(2) Find the transition matrix from $\mathcal{B}_0$ to $\mathcal{B}_1$.

(3) Given any $p(x) = a + bx + cx^2 + dx^3$ in $\mathbf{P}_3$, find the coordinates of $p(x)$ with respect to the basis $\mathcal{B}_1$.

Exercise 7 (14 pts)**.** Let A be a real $n \times n$-matrix such that $A^2 - 5A + 6I = \mathbf{O}$.

(1) Show that A is invertible.

(2) Use the equality $(x-2) - (x-3) = 1$ to show that

$$\mathbb{R}^n = \mathrm{Ker}(A - 2I) \oplus \mathrm{Ker}(A - 3I).$$

(3) Is A diagonalizable?

Exercise 8 (14 pts)**.** Find an orthogonal matrix that diagonalizes the symmetric matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Exercise 9 (14 pts)**.** Find the critical points of the function

$$f(x, y) = x^3 + 6xy + 3y^2 - 9x$$

and classify them as relative maxima, relative minima, or saddle points.

Exercise 10 (12 pts)**.** Express the quadratic equation

$$5x^2 + 5y^2 + 26xy - 2x - 34y = 43$$

in the matrix form

$$\mathbf{x}^T A\mathbf{x} + \mathbf{K}\mathbf{x} + f = 0,$$

where $\mathbf{x}^T A\mathbf{x}$ is the associated quadratic form and $\mathbf{K}$ is an appropriate matrix.

Identify the conic section represented by the given equation.