1. (a) Let $f : A \longrightarrow B$ be a function and let X, Y be subsets of A.

(i) Prove that $f(X \cap Y) \subseteq f(X) \cap f(Y)$.

(ii) Prove that if f is one-to-one, then $f(X) \cap f(Y) \subseteq f(X \cap Y)$.

(b) Let $C = \{4, 6, 8\}$ and $D = \{3, 4, 6\}$ and let g be the relation from C to D defined by

$$(a, b) \in g \text{ iff } 5 | (ab + 1).$$

Find the range of g. Is g a function? Justify.

2. (a) Prove that $|(0,1)| = |(0,9)|$

(b) Prove that $|\mathbb{Q} \cap (0,1)| = |\mathbb{N}|$

(c) State Schröder-Bernstein Theorem and use it to prove that $|\mathbb{R} - (1,2)| = |\mathbb{R}|$.

(d) Let A, B be sets where A is finite. Prove that if $A \times B$ is denumerable then B is denumerable.

3. (a) Find $\gcd(210, 792)$ and find integers x, y such that

$$\gcd(210, 792) = 210x + 792y.$$

(b) Let a, b, c be integers such that $a \neq 0$, $a|bc$ and $\gcd(a, b) = 1$. Prove that $a|c$.

(c) Let a, b, c, d be nonzero integers such that $a|c$ and $b|d$. Show that $\gcd(a, b) \leq \gcd(c, d)$.

(d) Without performing any division, show that $\gcd(12345, 6789)\,|123456789$.

4. (a) Let G be a group with identity e, and let $a, b \in G$. Prove that if $(ab)^3 = e$ then $(ba)^3 = e$.

(b) Let G, H, K be groups and let $g : G \longrightarrow H$, $h : H \longrightarrow K$ be isomorphisms. Prove that $h \circ g : G \longrightarrow K$ is an isomorphism.

(c) Use Lagrange's Theorem to determine all the subgroups of $(\mathbb{Z}_4, +)$.

(d) Let H, K be subgroups of a group G.

(i) Prove that $H \cap K$ is a subgroup of G.

(ii) Let $a, b \in G$. Prove that $ab^{-1} \in H$ iff $ba^{-1} \in H$.

5. (a) Let R be an equivalence relation on $\mathbb{Z}$. Find the domain and the range of R.

(b) Let S be the relation on $\mathbb{Q}$ defined by

$$aSb \text{ iff } a - b \in \mathbb{Z}.$$

Is S an equivalence relation? Justify.

(c) Let T be a relation on $\mathbb{Q}$ defined by

$$aTb \text{ iff } a - b \text{ is a nonnegative integer.}$$

Is $(\mathbb{Q}, T)$ a poset? Is it well-ordered? Justify your answers.