KFUPM/ Department of Mathematics & Statistics/181/MATH 210/Exam 2

Name: **ID#:** **Serial #:**

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1. [16pts] (a) Prove that there are no positive real numbers x, y such that $\sqrt{x} + 2\sqrt{y} = \sqrt{x + 4y}$.

(b) Prove that there are no integers a, b such that $a < b$ and $2a + 1 \geq 2b$.

Solution. (a) Assume for contradiction that there are positive real numbers x, y such that $\sqrt{x} + 2\sqrt{y} = \sqrt{x + 4y}$. Then $\left(\sqrt{x} + 2\sqrt{y}\right)^2 = x + 4y$, i.e. $x + 4y + 4\sqrt{xy} = x + 4y$.

Hence $xy = 0$, i.e. $x = 0$ or $y = 0$, a contradiction. ■

(b) Suppose on the contrary that there are integers a, b such that $a < b$ and $2a + 1 \geq 2b$.

Then $0 < 2(b - a) \leq 1$, which is impossible since $2(b - a)$ is an even integer and there are no even integers in the interval $(0, 1]$. ■

2. [24pts] (a) Prove that $2^n \geq (n + 1)^2$ for each integer $n \geq 6$.

(b) A sequence $\{a_n\}_{n \in \mathbb{N}}$ is defined recursively by:

$$a_1 = 2, \quad a_2 = 4, \quad a_{n+2} = 5a_{n+1} - 6a_n \text{ (for } n \geq 1).$$

Make a conjecture about a_n and then prove your claim.

Solution. (a) Let $P(n)$ be the statement: $2^n \geq (n + 1)^2$. Clearly $P(6)$ is true ($2^6 = 64 > 36 = 6^2$).

Suppose $P(k)$ is true for some integer $k \geq 6$ (i.e. $2^k \geq (k + 1)^2$).

We prove that $P(k + 1)$ is true (i.e. that $2^{k+1} \geq (k + 2)^2$).

We have $2^{k+1} = 2 \cdot 2^k \geq 2(k + 1)^2$ (by induction hypothesis).

If we prove that $2(k + 1)^2 \geq (k + 2)^2$, then we are done.

We have $2(k + 1)^2 - (k + 2)^2 = k^2 - 2 \geq 0$ since $k \geq 6$, hence $2(k + 1)^2 \geq (k + 2)^2$, as required. ■

(b) We have $a_3 = 5 \times 4 - 6 \times 2 = 2^3$, so we conjecture that $a_n = 2^n$ for each $n \geq 1$.

Let $P(n)$ be the statement: $a_n = 2^n$. Clearly $P(n)$ is true for $n = 1, 2$.

Let $k \geq 2$ be an integer and assume $P(h)$ is true for all integers h such that $1 \leq h \leq k$ (we are using strong induction and assuming that $P(1), P(2), \ldots, P(k)$ are true statements).

We prove that $P(k + 1)$ is true.

We have

$$\begin{aligned} a_{k+1} &= 5a_k - 6a_{k-1} = 5 \times 2^k - 6 \times 2^{k-1} \text{ (by induction hypothesis)} \\ &= 5 \times 2^k - 3 \times 2^k = 2^{k+1}, \text{ as required. } \blacksquare \end{aligned}$$

3. [20pts] (a) Let $\mathbb{R}^*$ denote the set of all nonzero real numbers. A relation R is defined on $\mathbb{R}^*$ by

$$xRy \text{ iff } x + y \neq 0.$$

Is R reflexive? symmetric? transitive? Justify your answers.

(b) In $\mathbb{Z}_8$, express $[15^{11}] + [11^{15}]$ as $[r]$ where $0 \leq r \leq 7$.

Solution. (a) For each nonzero real number x we have $x + x \neq 0$, hence xRx and so R is reflexive.
Suppose xRy for some nonzero real numbers x and y. Then $x + y \neq 0$, i.e. $y + x \neq 0$. Hence yRx and so R is symmetric.
R is not transitive: take $x = 1$, $y = 2$, $z = -1$. Then xRy and yRz but $x \not R z$.

(b) We have $15 \equiv -1 \pmod 8$ so $15^{11} \equiv -1 \pmod 8$
Also $11 \equiv 3 \pmod 8$, so $11^2 \equiv 9 \equiv 1 \pmod 8$.
We obtain $15^{11} + 11^{15} \equiv -1 + 11 \times (11^2)^7 \equiv 10 \equiv 2 \pmod 8$. Hence $[15^{11}] + [11^{15}] = [2]$ (in $\mathbb{Z}_8$).

4. [20pts] (a) Prove that the function $f : \mathbb{R} - \{1\} \longrightarrow \mathbb{R} - \{1\}$ given by $f(x) = \dfrac{x-3}{x-1}$ is a bijection and find its inverse function.

(b) For each of the following functions determine whether it is one-to-one or onto and justify your answers.
(i) $f : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{Q}$ given by $f(x, y) = \dfrac{x}{y}$.
(ii) $g : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ given by $g(x, y) = xy$.

Solution. (a) We have $f\left(\dfrac{x-3}{x-1}\right) = \dfrac{\dfrac{x-3}{x-1} - 3}{\dfrac{x-3}{x-1} - 1} = x$, i.e. $f \circ f = \mathrm{id}_{\mathbb{R}-\{1\}}$, the identity map on $\mathbb{R} - \{1\}$.
This shows that f is its own inverse and so f is bijective.
(We can also prove that f is a bijection by showing that it is injective and surjective.)

(b) (i) f is not one-to-one: $f(1, 1) = f(2, 2)$ but $(1, 1) \neq (2, 2)$.
f is not onto: There are no positive integers x, y such that $f(x, y) = 0$.

(ii) g is not one-to-one: $g(1, 2) = g(2, 1)$ but $(1, 2) \neq (2, 1)$.
g is onto: For each $n \in \mathbb{N}$ we have $g(1, n) = n$ i.e. n is an image under g.