Name: ID#: Serial #:

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1. [10pts] Mark each of the following statements as **true** or **false** and justify your answer.

(a) $\forall m \in \mathbb{N}$, $\exists n \in \mathbb{N}$, $m < n$. **True**: for each m in $\mathbb{N}$, take $n = m + 1$.

(b) $\exists m \in \mathbb{N}$, $\forall n \in \mathbb{N}$, $m < n$. **False**: if $m = 1$, then $n = 1$ does not satisfy $m < n$.

(c) $\exists m \in \mathbb{N}$, $\forall n \in \mathbb{N}$, $m|n$. **True**: if $m = 1$, then $1|n$ for each n in $\mathbb{N}$.

(d) $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$, $xy > x + 1$. **False**: if $x = 0$, then there is no y in $\mathbb{R}$ such that $xy > x + 1$.

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2. [10pts] Let $A = \{1, \{1\}\}$ and $B = \mathcal{P}(A)$.
(a) Verify that $A \cap B \in B$.

Solution. We have $B = \{\emptyset, \{1\}, \{\{1\}\}, A\}$. Hence $A \cap B = \{\{1\}\}$, which is an element of B.

(b) Find $|\mathcal{P}(A \times B)|$.

Solution. $|A \times B| = |A| \times |B| = 2 \times 4 = 8$. Hence $|\mathcal{P}(A \times B)| = 2^8$.

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3. [10pts] (a) Let $A = \{1, 2, 3\}$. Give an example of a subset B of A such that

$$(A \times A) - (B \times B) \neq (A - B) \times (A - B).$$

Solution. Let $B = \{1\}$ and let $a = (1, 2)$. Then $a \in A \times A$ but $a \notin B \times B$ ($\because 2 \notin B$), so $a \in (A \times A) - (B \times B)$. However, $1 \notin A - B$, so $a \notin (A - B) \times (A - B)$. This shows that $(A \times A) - (B \times B) \neq (A - B) \times (A - B)$.

(b) Prove that if C, D are sets, then $(C \times D) \cap (D \times C) = (C \cap D) \times (C \cap D)$.

Proof. Let $(x, y) \in (C \times D) \cap (D \times C)$. Then $(x, y) \in C \times D$ and $(x, y) \in D \times C$, so $x \in C \cap D$ and $y \in C \cap D$, i.e. $(x, y) \in (C \cap D) \times (C \cap D)$. This proves $(C \times D) \cap (D \times C) \subseteq (C \cap D) \times (C \cap D)$. For the reverse inclusion, let $(x, y) \in (C \cap D) \times (C \cap D)$. Then $x \in C \cap D$ and $y \in C \cap D$, so that $(x, y) \in C \times D$ and $(x, y) \in (D \times C)$, i.e. $(x, y) \in (C \times D) \cap (D \times C)$. This proves $(C \cap D) \times (C \cap D) \subseteq (C \times D) \cap (D \times C)$ and the proof is complete. ■

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4. [10pts] Let $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Prove the following statements.

(a) If $|1 + n| + |1 - n| \leq 1$, then $|n^4 - 1| < 16$.

Proof. Since $n \geq 1$, $|1 + n| \geq 2$, hence $|1 + n| + |1 - n| \geq 2$. This means that the statement "$|1 + n| + |1 - n| \leq 1$" is false and so the implication to prove is vacuously true. ■

(b) If $x^2 > 4$, then $|1 + x| + |1 - x| \geq 2$.

Proof. By the triangle inequality, $|1 + x| + |1 - x| \geq |(1 + x) + (1 - x)| = 2$. Hence the statement "$|1 + x| + |1 - x| \geq 2$" is true and so the implication to prove is trivially true. ■

5. [10pts] Let x, y be integers. Prove the following statements.

(a) If x and y are odd, then $4 \mid \left((x+y)^2 + (x-y)^2\right)$.

Proof. Let $x = 2h+1$, $y = 2k+1$ $(h, k \in \mathbb{Z})$. Then $(x+y)^2 + (x-y)^2 = 2\left(x^2+y^2\right) = 2\left(4h^2+4h+1+4k^2+4k+1\right) = 4\left(2h^2+2h+2k^2+2k+1\right)$. Since $2h^2+2h+2k^2+2k+1 \in \mathbb{Z}$, it follows that $4 \mid \left((x+y)^2 + (x-y)^2\right)$. ■

(b) If $4 \mid \left(x^2+2y^2\right)$, then x and y are both odd or both even.

Proof.

- We can use a direct proof:

Suppose $4 \mid \left(x^2+2y^2\right)$, then $x^2+2y^2 = 4k$ for some integer k. Hence $x^2 = 2\left(2k-y^2\right)$, which is even. Therefore x must be even, say $x = 2h$ for some integer h. We then get $4h^2+2y^2 = 4k$, i.e. $y^2 = 2\left(k-2h^2\right)$, which is even. Hence y^2 is even and so y is even as well. This means x and y are both even and so they have the same parity.

- We can also use the contrapositive:

Suppose x and y have opposite parity. We have two cases:
(i) Case x is even and y is odd: In this case $x = 2h$, $y = 2k+1$ for some integers h, k. Hence $x^2+2y^2 = 4h^2+2\left(4k^2+4k+1\right) = 2\left(2h^2+4k^2+4k+1\right)$, and since $2h^2+4k^2+4k+1 = 2\left(h^2+2k^2+2k\right)+1$ is odd (as $h^2+2k^2+2k \in \mathbb{Z}$), we deduce that x^2+2y^2 is not divisible by 4 (but it is even).
(ii) Case x is odd and y is even: In this case $x = 2h+1$, $y = 2k$ for some integers h, k. Hence $x^2+2y^2 = 4\left(h^2+h+k^2\right)+1$, which clearly is odd (as $h^2+h+k^2 \in \mathbb{Z}$) so that $4 \nmid \left(x^2+2y^2\right)$. ■

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6. [10pts] Let $a, b \in \mathbb{Z}$. Prove the following statements.

(a) If $a \equiv 1 \pmod 2$ and $b \equiv 2 \pmod 3$, then $3a - 2b \equiv 5 \pmod 6$

Proof. $3a \equiv 3 \pmod 6$ and $2b \equiv 4 \pmod 6$, so $3a - 2b \equiv -1 \equiv 5 \pmod 6$. ■

(b)If $a \not\equiv 2 \pmod 4$, then $a^3 \equiv a \pmod 4$.

Proof. We have 3 cases.
Case $a \equiv 0 \pmod 4$: then $a^3 \equiv 0 \equiv a \pmod 4$.
Case $a \equiv 1 \pmod 4$: then $a^3 \equiv 1 \equiv a \pmod 4$.
Case $a \equiv 3 \pmod 4$: then $a \equiv -1 \pmod 4$, so $a^3 \equiv -1 \equiv a \pmod 4$. ■