KFUPM – Department of Mathematics & Statistics

**MATH202 – Term 181**

Quiz 3B (Duration: 25 Minutes)

Name: Solution ID #: ______ Section: 03

**Q1** (*3 Pts*) Without solving the DE, show that the set of functions $\{x^2, x^2 \ln x\}$ form a fundamental set of solutions of the DE $x^2y'' - 3xy' + 4y = 0,\ x > 0$.

For $y_1 = x^2,\quad y_1' = 2x,\ y_1'' = 2$

$\Rightarrow \quad x^2(2) - 3x(2x) + 4x^2$

$= 2x^2 - 6x^2 + 4x^2 = 0$

So $y_1$ is a Solution

For $y_2 = x^2 \ln x,\ y_2' = 2x\ln x + x$

$y_2'' = 2\ln x + 3$

$\Rightarrow \quad x^2(2\ln x + 3) - 3x(2x\ln x + x) + 4x^2 \ln x$

$= 2x^2\ln x + 3x^2 - 6x^2\ln x + 3x^2 + 4x^2\ln x = 0$

So, $y_2$ is a Solution.

$$W = \begin{vmatrix} x^2 & x^2\ln x \\ 2x & 2x\ln x + x \end{vmatrix}$$

$= 2x^3\ln x + x^3 - 2x^3 \ln x$

$= x^3 \neq 0$

So The given set is linearly independent.

Thus, the set forms a fundamental set of solution of the DE.

**Q2** (*3.5 Pts*) Let $L$ be a linear differential operator. Given that $y_1 = 1 + x,\ y_2 = \cos x,\ y_3 = x - e^{-x}$ are particular solutions of the nonhomogeneous equations

$$L(y) = -4 + 2x, \qquad L(y) = \sin^2 x, \qquad L(y) = xe^x,$$

Respectively, find a particular solution of the DE $L(y) = xe^{1+x} + 3 - x - \cos 2x$.

Let $g_1(x) = -4 + 2x$

$g_2(x) = \sin^2 x$

$g_3(x) = xe^x$

and

$g(x) = xe^{1+x} + 3 - x - \cos 2x$

$= exe^x + 2 - x + 1 - \cos 2x$

$= exe^x - \frac{1}{2}(-4+2x) + 2\left[\frac{1}{2}(1 - \cos 2x)\right]$

$= exe^x - \frac{1}{2}(-4+2x) + 2\left[\frac{1}{2}(1 - \cos 2x)\right]$

$= exe^x - \frac{1}{2}(-4 + 2x) + 2\sin^2 x$

So

$g(x) = e\,g_3(x) - \frac{1}{2}g_1(x) + 2g_2(x)$

Thus,

$y_p = ey_3 - \frac{1}{2}y_1 + 2y_2$

$= ex - e^{1-x} - \frac{1}{2}(1+x) + 2\cos x$

**Q3.** (*3.5 pts*) Given that $y_1 = (1+x^2)^{1/2}$ is a solution of the DE $(1+x^2)y'' + xy' - y = 0$. Find a second solution $y_2$ of the given differential equation.

Solution:

The Standard form is

$$y'' + \frac{x}{1+x^2}y' - \frac{y}{1+x^2} = 0$$

$$P(x) = \frac{x}{1+x^2}$$

$$y_2(x) = y_1(x)\int \frac{e^{-\int P(x)\,dx}}{[y_1(x)]^2}\,dx = (1+x^2)^{1/2}\int \frac{e^{-\int \frac{x}{1+x^2}dx}}{1+x^2}\,dx$$

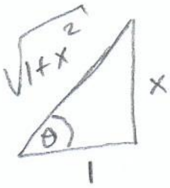
$$= (1+x^2)^{1/2}\int \frac{e^{-\frac{1}{2}\ln(1+x^2)}}{1+x^2}\,dx$$

$$= (1+x^2)^{1/2}\int \frac{(1+x^2)^{-1/2}}{1+x^2}\,dx = (1+x^2)^{1/2}\int (1+x^2)^{-3/2}\,dx$$

Put $x = \tan\theta$

$dx = \sec^2\theta\, d\theta$, so $\int (1+x^2)^{-3/2}dx = \int (1+\tan^2\theta)^{-3/2}\sec^2\theta\, d\theta$

$$= \int (\sec^2\theta)^{-3/2}\sec^2\theta\, d\theta = \int \frac{1}{\sec\theta}\,d\theta = \int \cos\theta\, d\theta = \sin\theta$$




$$= \frac{x}{(1+x^2)^{1/2}}$$

Thus,

$$y_2(x) = (1+x^2)^{1/2}\frac{x}{(1+x^2)^{1/2}}$$

$$= x$$

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Quiz 3A (Duration: 25 Minutes)

Name: Solution ID #: ______ Section: 03

**Q1** (*3 Pts*) Let $L$ be a linear differential operator. Given that $y_1 = 1 + x,\ y_2 = \cos x,\ y_3 = x - e^{-x}$ are particular solutions of the nonhomogeneous equations

$$L(y) = 4 + 2x, \qquad L(y) = x\sin x, \qquad L(y) = xe^{x},$$

Respectively, find a particular solution of the DE $L(y) = -2 + x(-1 + \sin x + 3e^{x-1})$.

Let $g_1(x) = 4 + 2x$

$g_2(x) = x \sin x$

$g_3(x) = xe^{x}$.

and

$g(x) = -2 + x(-1 + \sin x + 3e^{x-1})$

$= -2 - x + x\sin x + \frac{3}{e} e^{x}$

$= -\frac{1}{2}(4 + 2x) + x\sin x + \frac{3}{e} e^{x}$

$= -\frac{1}{2} g_1(x) + g_2(x) + \frac{3}{e} g_3(x)$

By the Superposition principle, we obtain

$y_p = -\frac{1}{2} y_1 + y_2 + \frac{3}{e} y_3$

$= -\frac{1}{2}(1+x) + \cos x + \frac{3}{e}(x - e^{-x})$

**Q2** (*3.5 Pts*) Without solving the DE, show that the set of functions $\{x, x^{-1}\}$ form a fundamental set of solutions of the DE $x^2y'' + xy' - y = 0,\ x > 0$.

For $y_1 = x$, $y' = 1$, $y'' = 0$

$\Rightarrow x^2(0) + x(1) - x =$

$x - x = 0$

So $y_1$ is a solution.

For $y_2 = x^{-1}$, $y_2' = -x^{-2}$

$y_2'' = 2x^{-3}$.

$\Rightarrow x^2(2x^{-3}) + x(-x^{-2}) - x^{-1}$

$= 2x^{-1} - x^{-1} - x^{-1} = 0$

So $y_2$ is a solution

$W = \begin{vmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{vmatrix} = -x^{-1} - x^{-1}$

$= -2x^{-1} \neq 0$

So, the given set is linearly independent.

Thus, the set forms a fundamental set of solutions of the DE.

**Q3.** (*3.5 pts*) Given that $y_1 = \frac{\sin x}{\sqrt{x}}$ is a solution of the DE $4x^2y'' + 4xy' + (4x^2 - 1)y = 0,\ x > 0.$
Find the general solution of the given differential equation.

The Standard form of the DE is

$$y'' + \frac{y'}{x} + \left(1 - \frac{1}{4x^2}\right)y = 0, \qquad x > 0.$$

$$P(x) = \frac{1}{x}.$$

The Second Solution is

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{[y_1(x)]^2}\,dx$$

$$= \frac{\sin x}{\sqrt{x}} \int \frac{e^{-\int \frac{1}{x}dx}}{\sin^2 x / x}\,dx$$

$$= \frac{\sin x}{\sqrt{x}} \int \frac{e^{-\ln x}}{\sin^2 x / x}\,dx$$

$$= \frac{\sin x}{\sqrt{x}} \int \frac{x^{-1}}{x^{-1}\sin^2 x}\,dx$$

$$= \frac{\sin x}{\sqrt{x}} \int \frac{1}{\sin^2 x}\,dx$$

$$= \frac{\sin x}{\sqrt{x}} \int \csc^2 x\,dx$$

$$= \frac{\sin x}{\sqrt{x}}(-\cot x) = \frac{\sin x}{\sqrt{x}}\left(-\frac{\cos x}{\sin x}\right)$$

$$= -\frac{\cos x}{\sqrt{x}}$$

We can write $y_2(x) = \frac{\cos x}{\sqrt{x}}$