1 (10 points) Classify the singular points of

$$x(x^2-x-6)^2y'' + (x-3)y' + x(x+2)y = 0.$$

1 $$y'' + \frac{(x-3)}{x(x-3)^2(x+2)^2}\,y' + \frac{(x)(x+2)}{x(x-3)^2(x+2)^2}\,y = 0$$

singular points are: $x=0$, $x=3$ and $x=-2$ 1

$$P(x) = \frac{1}{x(x+2)^2(x-3)^2}, \quad Q(x) = \frac{1}{(x-3)^2(x+2)}$$

for $x=0$ $$p(x) = \frac{1}{(x+2)^2(x-3)} \;\; 1, \quad q(x) = \frac{x}{(x-3)^2(x+2)} \;\; 1$$

are analytic at $x=0 \Rightarrow$ regular sing. point 1

for $x=3 \Rightarrow$ $$p(x) = \frac{1}{x(x+2)^2} \;\; 1, \quad q(x) = \frac{1}{(x+2)} \;\; 1$$

are analytic at $x=3 \Rightarrow$ reg sing point 1

for $x=-2$, $$p(x) = \frac{1}{x(x+2)(x-3)} \;\; 1$$

is not analytic at $x=-2$ 1

So, $x=0$ and $x=3$ are regular singular points

and

$x=-2$ is an irregular singular point

2. (9 pts) Find a nonhomogenous second order linear differential equation with constant coefficients with $y_p = \frac{1}{10}e^{6x}$, as a particular solution and $y_1 = e^{-x}\cos x$, is a solution of the associated homogenous differential equation.

$y_1 = e^{-x}\cos x$ is a solution $\Rightarrow -1 \pm i$ are solutions (1) of the auxiliary eqn. which is given as:

$(m+1-i)(m+1+i) = m^2 + 2m + 2$ (1)

The associated homog. eqn. is

$$y'' + 2y' + 2y = 0 \quad (2)$$

if the non-homog eqn is: $y'' + 2y' + 2y = g(x)$

$y_p = \frac{1}{10}e^{6x}$, $y_p' = \frac{3}{5}e^{6x}$, $y_p'' = \frac{18}{5}e^{6x}$ (1)

$$\Rightarrow \frac{18}{5}e^{6x} + \frac{6}{5}e^{6x} + \frac{1}{5}e^{6x} = g(x) \quad (1)$$

$$\Rightarrow g(x) = 5e^{6x} \quad (1)$$

$$\Rightarrow y'' + 2y' + 2y = 5e^{6x} \quad (2)$$

(10 pts) 3. Solve the differential equation

$y'' - y = xe^x + 5$; $\quad m^2 - 1 = 0 \Rightarrow m = \pm 1$ (1)

$y_c = c_1 e^{x} + c_2 e^{-x}$ (1)

annihilator of the RHS is: $D(D-1)^2$ (1)

applying to the eqn.: $D(D-1)^2(D^2-1)y = 0$

The roots of the aux. eqn. are

$m_1 = -1, \quad m_{2,3,4} = 1, \quad m_5 = 0$ (1)

$\Rightarrow y = c_1 e^{x} + c_2 e^{-x} + c_3 x e^{x} + c_4 x^2 e^{x} + c_5$

$\Rightarrow y_p = Axe^{x} + Bx^2e^{x} + C$ (1)

$y_p' = Axe^{x} + Ae^{x} + Bx^2e^{x} + 2Bxe^{x}$

$y_p'' = Axe^{x} + 2Ae^{x} + Bx^2e^{x} + 4Bxe^{x} + 2Be^{x}$

$y_p'' - y_p = xe^{x} + 5 \Rightarrow (2A+2B)e^{x} + 4Bxe^{x} - C = xe^{x} + 5$

(1) $\boxed{C = -5}$ $\quad 4B = 1, \quad \boxed{B = \frac{1}{4}}$ (1)

$2A + 2B = 0 \Rightarrow \boxed{A = -1/4}$ (1)

$y_p = -\frac{1}{4} x e^{x} + \frac{1}{4} x^2 e^{x} - 5$ (1)

general solution $y = y_c + y_p$ (1)

Method 2 : Variation of parameters

$$y_c = c_1 \underbrace{e^{x}}_{y_1} + c_2 \underbrace{e^{-x}}_{y_2}$$

$$W = \begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix} = -2 \qquad (2)$$

$$y_p = u_1 y_1 + u_2 y_2 , \qquad (2)$$

$$u_1' = -\frac{e^{-x}}{-2}(xe^{x}+5) = \frac{1}{2}(x+5e^{-x})$$

$$\Rightarrow u_1 = \frac{1}{2}\int (x+5e^{-x})\,dx$$

$$= \frac{1}{2}\left(\frac{x^2}{2} - 5e^{-x}\right) \qquad (2)$$

$$u_2' = \frac{e^{x}}{-2}(xe^{x}+5) = -\frac{1}{2}(xe^{2x}+5e^{x})$$

$$\Rightarrow u_2 = -\frac{1}{2}\int (xe^{2x}+5e^{x})\,dx$$

$$= -\frac{1}{2}\left[\left(\frac{x}{2}-\frac{1}{4}\right)e^{2x}+5e^{x}\right] \qquad (2)$$

$$\Rightarrow x_p = \frac{1}{2}\left(\frac{x^2}{2}-5e^{-x}\right)e^{x} - \frac{1}{2}\left[\left(\frac{x}{2}-\frac{1}{4}\right)e^{2x}+5e^{x}\right]e^{-x}$$

$$= \left(\frac{x^2}{4} - \frac{x}{4} + \frac{1}{8}\right)e^{x} - 5 \qquad (2)$$

4. (10 points) Solve the linear system: $X' = \begin{bmatrix} 1 & -3 & 2 \\ 0 & -1 & 0 \\ 0 & -1 & -2 \end{bmatrix} X.$

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & -3 & 2 \\ 0 & -1-\lambda & 0 \\ 0 & -1 & -2-\lambda \end{vmatrix} = (1-\lambda)\ (1+\lambda)(2+\lambda) = 0 \quad 1 \quad 1$$

$\lambda = 1, -1, -2$ 1

$\boxed{\lambda = 1}$: $\begin{bmatrix} 0 & -3 & 2 \\ 0 & -2 & 0 \\ 0 & -1 & -3 \end{bmatrix} \Rightarrow V_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{t}$ 1 1

$\boxed{\lambda = -1}$ $\begin{bmatrix} 2 & -3 & 2 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $x_1 = -5/2\, x_3$, $x_2 = -x_3$, $x_3 = -2$

$\Rightarrow V_2 = \begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix} \Rightarrow X_2 = \begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix} e^{-t}$ 1 1

$\boxed{\lambda = -2}$ $\begin{bmatrix} 3 & -3 & 2 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$ $x_1 = \frac{-2x_3}{3}$, $x_3 = 3$, $x_2 = 0$ $\Rightarrow V_3 = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}; X_3 = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} e^{-2t}$ 1 1

Solut: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix} e^{t} + c_3 \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} e^{-2t}$ 1

5. (5 pts) Find a number k so that $X_p = k\begin{bmatrix}1\\2\end{bmatrix}t + k\begin{bmatrix}0\\-1\end{bmatrix}$ is a solution of the linear system

$$X' = \begin{bmatrix}2 & -1\\3 & -2\end{bmatrix}X + \begin{bmatrix}0\\4\end{bmatrix}t$$

$$X_p = \begin{pmatrix}kt\\2kt\end{pmatrix} + \begin{pmatrix}0\\-k\end{pmatrix} = \begin{pmatrix}kt\\2kt-k\end{pmatrix} \quad 1$$

$$X_p' = \begin{pmatrix}k\\2k\end{pmatrix} \quad 1$$ Sub. into the system

$$\begin{pmatrix}k\\2k\end{pmatrix} = \begin{pmatrix}2 & -1\\3 & -2\end{pmatrix}\begin{pmatrix}kt\\2kt-k\end{pmatrix} + \begin{pmatrix}0\\4t\end{pmatrix} \quad 1$$

$$\begin{pmatrix}k\\2k\end{pmatrix} = \begin{pmatrix}2kt-2kt+k\\3kt-4kt+2k\end{pmatrix} + \begin{pmatrix}0\\4t\end{pmatrix}$$

$$\begin{pmatrix}k\\2k\end{pmatrix} = \begin{pmatrix}k\\-kt+2k+4t\end{pmatrix} = \begin{pmatrix}k\\(4-k)t+2k\end{pmatrix} \quad 1$$

$$4-k=0 \Rightarrow k=4 \quad 1$$

6. (10 pts) Solve the initial-value problem

$$X' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} X, \quad X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 4\lambda + 4 = 0 \quad 1$$

$$\Rightarrow \lambda = 2, 2 \quad 1 \text{ repeated.}$$

$\underline{\lambda=2}$: $(A-2I)K = 0 \Rightarrow \left(\begin{array}{cc|c} -1 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right) \Rightarrow x+y=0$

$1 \quad 1 \quad K\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad 1$

$(A-2I)P = K \Rightarrow \left(\begin{array}{cc|c} -1 & -1 & 1 \\ 1 & 1 & -1 \end{array}\right) \Rightarrow P\begin{pmatrix} -1 \\ 0 \end{pmatrix}$

1

$\underline{\text{Thus:}}$ $X_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} \quad 1$

$$X_2 = \left[\begin{pmatrix} t \\ -t \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right] e^{2t} \quad 1$$

general solution

$$X = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} t-1 \\ -t \end{pmatrix} e^{2t} \quad 1 \qquad t \in (-\infty, \infty)$$

$$X(0) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad c_1 - c_2 = 1 \qquad c_2 = -2$$

$$-c_1 = 1 \quad \Rightarrow \quad c_1 = -1$$

$$X = -\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} - 2\begin{pmatrix} t-1 \\ -t \end{pmatrix} e^{2t}, \quad t \in (-\infty, \infty) \quad 1$$

7. (14 pts) Solve the system $X' = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} X.$

1

$$P_A(\lambda) = \begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 1-\lambda \end{vmatrix} = (2-\lambda)\left[(1-\lambda)^2+1\right] \quad 1$$

$$\Rightarrow \lambda = 2, \quad \lambda = 1+i, \quad \bar{\lambda} = 1-i \quad 2$$

$$\lambda=2: \left(\begin{array}{ccc|c} -1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right) \Rightarrow v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad 1 \quad \Rightarrow X_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{2t} \quad 1$$

1 $\qquad\qquad$ 2

$$\lambda = 1+i: \left(\begin{array}{ccc|c} -i & 0 & -1 & 0 \\ 1 & 1-i & 1 & 0 \\ 1 & 0 & -i & 0 \end{array}\right) \Rightarrow K = \begin{pmatrix} a \\ -a \\ -ia \end{pmatrix} \Rightarrow K = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -i \end{pmatrix}$$

$$X_1 = e^t \cos t \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - e^t \sin t \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} e^t \cos t \\ -e^t \cos t \\ e^t \sin t \end{pmatrix} \quad 2$$

$$X_2 = e^t \cos t \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + e^t \sin t \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^t \sin t \\ -e^t \sin t \\ -e^t \cos t \end{pmatrix} \quad 2$$

$$X = \begin{pmatrix} (c_2 \cos t + c_3 \sin t)\, e^t \\ c_1 e^{2t} - (c_2 \cos t + c_3 \sin t)\, e^t \\ (c_2 \sin t - c_3 \cos t)\, e^t \end{pmatrix} \quad 1$$

8. (14 pts) Find a recurance relation for the power series solutions of the differential equation

$$(1-x^2)y'' - y = 0,$$

about the ordinary point $x_0 = 0$. (Do not solve the equation)

$$y = \sum_{n=0}^{\infty} c_n x^n, \quad (1) \quad |x| < 1$$

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}; \quad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} \quad (1)$$

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^n = 0 \quad (1)$$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) c_n x^n - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{k=0}^{\infty} (k+1)(k+2) c_{k+2} x^k - \sum_{k=2}^{\infty} k(k-1) c_k x^k - \sum_{k=0}^{\infty} c_k x^k = 0 \quad (1)$$

$$2c_2 - c_0 + (6c_3 - c_1)x + \sum_{k=2}^{\infty} \left[(k+1)(k+2)c_{k+2} - (1+k(k-1))c_k\right] x^k = 0 \quad (1)$$

(1) $c_2 = c_0/2$, $c_3 = c_1/6$ (1)

(1) $c_{k+2} = \dfrac{1+k(k-1)}{(k+1)(k+2)} c_k \qquad k = 2, 3, 4, \ldots$

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots$$

$$= c_0 + c_1 x + \frac{c_0}{2} x^2 + \frac{c_1}{6} x^3 + \frac{c_0}{8} x^4 + \frac{7}{120} c_1 x^5 + \cdots$$

$$= c_0 \underbrace{\left(1 + \frac{x^2}{2} + \frac{x^4}{8} + \cdots\right)}_{y_1 \ (3)} + c_1 \underbrace{\left(x + \frac{x^3}{6} + \frac{7x^5}{120} + \cdots\right)}_{y_2 \ (3)}$$

9. (12 points) Find the first nonzero terms of the power series solution of the differential equation $xy'' - 2y' + 2y = 0$ corresponding to the largest root of the indicial equation.

Put $y = \sum_{n=0}^{\infty} c_n x^{n+r}$, $y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1}$

and $y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$ (2)

Sub. in the equ. (1)

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-1} - \sum_{n=0}^{\infty} 2(n+r) c_n x^{n+r-1} + \sum_{n=0}^{\infty} 2 c_n x^{n+r} = 0$$

$$x^r \Big[\underbrace{\sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n-1}}_{\substack{k=n-1 \\ n=k+1}} - \underbrace{\sum_{n=0}^{\infty} 2(n+r) c_n x^{n-1}}_{\substack{k=n-1 \\ n=k+1}} + \underbrace{\sum_{n=0}^{\infty} 2 c_n x^{n}}_{k=n} \Big] = 0 \quad (2)$$

$$\sum_{k=-1}^{\infty} (k+r+1)(k+r) c_{k+1} x^k - \sum_{k=-1}^{\infty} 2(k+r+1) c_{k+1} x^k + \sum_{k=0}^{\infty} 2 c_k x^k = 0$$

$$r(r-1) c_0 x^{-1} - 2r c_0 x^{-1} + \sum_{k=0}^{\infty} \Big[(k+r+1) c_{k+1} [k+r-2] + 2c_k \Big] x^k = 0$$

$$(r^2 - r - 2r) c_0 x^{-1} + \sum_{k=0}^{\infty} \Big[(k+r+1)(k+r-2) c_{k+1} + 2c_k \Big] x^k = 0 \quad (1)$$

The indicial eqn is $r^2 - 3r = 0$, (1) $r_1 = 3$, (1) $r_2 = 0$

and the solu corresponding to the largest root is

$$y = \sum_{n=0}^{\infty} c_n x^{n+3}$$

$$c_{k+1} = \frac{-2c_k}{(k+4)(k+1)} \quad (1)$$

c_0 (1)

$c_1 = -\frac{c_0}{2}$

$c_2 = \frac{c_0}{10}$ (1)

$$y = c_0 \left(x^3 - \frac{x^4}{2} + \frac{x^5}{10} + \cdots \right)$$

(1)

10. (10 pts) Let $X = \begin{pmatrix} 2e^{7t} & -2e^{-5t} \\ e^{7t} & e^{-5t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ be the general solution of the homogeneous system $X' = AX$.

Find a particular solution of $X' = AX + \begin{pmatrix} e^{7t} \\ 0 \end{pmatrix}$.

<u>Answer</u>

$$\phi = \begin{pmatrix} 2e^{7t} & -2e^{-5t} \\ e^{7t} & e^{-5t} \end{pmatrix} \quad 1 \quad , \quad \det\phi = 2e^{2t} + 2e^{2t} = 4e^{2t} \quad 1$$

$$\phi^{-1} = \frac{e^{-2t}}{4}\begin{pmatrix} e^{-5t} & 2e^{-5t} \\ -e^{7t} & 2e^{7t} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} e^{-7t} & 2e^{-7t} \\ -e^{5t} & 2e^{5t} \end{pmatrix} \quad 2$$

$$U = \int \phi^{-1} F = \frac{1}{4}\int \begin{pmatrix} e^{-7t} & 2e^{-7t} \\ -e^{5t} & 2e^{5t} \end{pmatrix}\begin{pmatrix} e^{7t} \\ 0 \end{pmatrix} dt \quad 1$$

$$= \frac{1}{4}\int \begin{pmatrix} 1 \\ -e^{12t} \end{pmatrix} \quad 1$$

$$= \frac{1}{4}\begin{pmatrix} t \\ -\frac{e^{12t}}{12} \end{pmatrix} \quad 2$$

$$\Rightarrow X_p = \phi U = \frac{1}{4}\begin{pmatrix} 2e^{7t} & -2e^{-5t} \\ e^{7t} & e^{-5t} \end{pmatrix}\begin{pmatrix} t \\ -\frac{e^{12t}}{12} \end{pmatrix} \quad 1$$

$$= \frac{1}{4}\begin{pmatrix} 2te^{7t} + \frac{1}{6}e^{7t} \\ te^{7t} - \frac{1}{12}e^{7t} \end{pmatrix} \quad 1$$

11. (6 pts) $\frac{dx}{dt} = 2x - y$

$\frac{dy}{dt} = 4x - 2y$

$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ 1

Solution: $A = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$ 1 $\Rightarrow A^2 = 0$ 1

$$\exp[At] = \begin{bmatrix} 1+2t & -t \\ 4t & 1-2t \end{bmatrix}$$ 2

$$X(t) = e^{At} \cdot C$$

$$X(t) = \begin{bmatrix} 1+2t & -t \\ 4t & 1-2t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$ 1

12. (10 pts) Use a suitable substitution to change the following DE to linear DE or separable (Do not the solve the new equation.)

(a) $x^2y' = 2xy + 3y^3$. (6 points)

(b) $xy' = x\cos\left(\frac{y}{x}\right) + y$.

(a) $y' - \frac{2}{x}y = \frac{3}{x^2}y^3$ (1)

Bernoulli DE with $n=3 \Rightarrow u = y^{-2}$ (1)

or $y = u^{-1/2}$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -\frac{1}{2}u^{-3/2}\frac{du}{dx} \quad (1)$$

Sub. in DE

$$-\frac{1}{2}u^{-3/2}\frac{du}{dx} - \frac{2}{x}u^{-1/2} = \frac{3}{x^2}u^{-3/2} \quad (1)$$

$$\Rightarrow \frac{du}{dx} + \frac{4}{x}u = -\frac{6}{x^2} \quad (1)$$

(b) $y' = \cos\left(\frac{y}{x}\right) + \frac{y}{x}$ Homog. DE (1)

Let $v = \frac{y}{x}$ (1) $\Rightarrow y = vx \Rightarrow y' = v + x\frac{dv}{dx}$ (1)

Sub in DE $\Rightarrow v + x\frac{dv}{dx} = \cos(v) + v$ (1)

$$x\frac{dv}{dx} = \cos(v)$$

$$\frac{dv}{\cos(v)} = \frac{dx}{x} \quad \text{Seperable (1)}$$

Show that the equation is exact and solve it.

B (10 points)

$$(e^x\cos(y)+1)y' = 2x - e^x\sin(y), \quad y(1)=0.$$

$$[e^x\cos y + 1]\,dy = [2x - e^x \sin y]\,dx$$

$$\underbrace{[e^x \sin y - 2x]}_{M}\,dx + \underbrace{[e^x\cos y + 1]}_{N}\,dy = 0 \quad 1$$

$M_y = e^x \cos y$

$N_x = e^x \cos y$ exact DE

$f_x = e^x \sin y - 2x \,, \quad f_y = e^x\cos y + 1 \quad 2$

$\Rightarrow f = e^x \sin y - x^2 + g(y) \quad 2$

$\Rightarrow f_y = e^x\cos y + g'(y) = e^x \cos y + 1 \quad 1$

$g(y) = y \quad 1$

$\Rightarrow f(x,y) = e^x \sin y - x^2 + y = C \quad 1$

$y(1)=0 \Rightarrow e \sin 0 - 1 + 0 = C$

$\Rightarrow C = -1 \quad 1$

Thus, the particular sol. is

$$e^x \sin y - x^2 + y = -1 \quad 1$$

14. (10 points) Solve the differential equation

$$x^3y''' + 4x^2y'' + 3xy' + y = 0. \quad , \quad x > 0$$

Let 1 $y = x^n \Rightarrow y' = n x^{n-1}, \; y'' = n(n-1)x^{n-2},$
$y^{(3)} = n(n-1)(n-2)x^{n-3}$

So the DE becomes 1

$n(n-1)(n-2)x^n + 4n(n-1)x^n + 3nx^n + x^n = 0$

$n(n^2-3n+2)x^n + 4(n^2-n)x^n + 3n + x^n = 0$

aux. eqn. $\Rightarrow n^3 - 3n^2 + 2n + 4n^2 - 4n + 3n + 1 = 0$

$n^3 + n^2 + n + 1 = 0$ 3

$n^2(n+1) + (n+1) = 0$

$(n+1)(n^2+1) = 0$ 1

$n = -1, \; \pm i$ 2

So the sol. of the DE is

$y = c_1 x^{-1} + c_2 \cos(\ln x) + c_3 \sin(\ln x)$

2