1. (7 points)

(a) Verify that $y = c_1 x + c_2 x \ln x$ is a solution of $x^2 y'' - xy' + y = 0$.

(b) Verify that the BVP

$$x^2 y'' - xy' + y = 0, \quad y(1) = 3,\ y'(e^{-1}) = 3.$$

has a one parameter family of solution.

(a) $y' = c_1 + c_2(\ln x + 1)$ 1

$y'' = \dfrac{c_2}{x}$ 1

sub in the DE $\Rightarrow$ $x^2 y'' - x y' + y = 0$ 1

LHS: $x^2\left(\dfrac{c_2}{x}\right) - x\left[c_1 + c_2 \ln x + c_2\right] + c_1 x + c_2 x \ln x$

$= c_2 x - c_1 x - c_2 x \ln x - c_2 x + c_1 x + c_2 x \ln x = 0$

$\Rightarrow$ LHS = RHS $\Rightarrow$ solution 1

(b) $y = c_1 x + c_2 x \ln x$, $y(1) = 3$, $y'(e^{-1}) = 3$

$y(1) = 3 \Rightarrow 3 = c_1 + c_2(0) \Rightarrow \boxed{c_1 = 1}$ 1

$y'(e^{-1}) = 3 \Rightarrow 3 = c_1 + c_2(\ln e^{-1} + 1)$

$\Rightarrow \boxed{3 = c_1}$ 1

Solution $\Rightarrow$ $y = 3x + c_2 x \ln x$ 1

one-family parameter of sol.

2. (10 points) Find the values of B so that the IVP

$$\frac{dy}{dx} = \frac{\sqrt{y-2x}}{e^{1/y}}, \quad y(-1) = B$$

has a unique solution.

$$f(x,y) = \frac{\sqrt{y-2x}}{e^{1/y}} = e^{-1/y}\sqrt{y-2x} \quad (1)$$

$$\frac{\partial f}{\partial y} = e^{-1/y}\frac{1}{y^2}\sqrt{y-2x} + e^{-1/y}\left(\frac{1}{2\sqrt{y-2x}}\right) \quad (1)$$

$$= \frac{\sqrt{y-2x}}{y^2 e^{1/y}} + \frac{1}{2e^{1/y}\sqrt{y-2x}} \quad (1)$$

f and $\frac{\partial f}{\partial y}$ are continuous if

$$y > 2x, \quad y \neq 0 \quad (2)$$

From the initial condition $y(-1) = B$

we obtain $x = -1$, $y = B$

Thus $B > 2(-1) = -2$ and $B \neq 0$ (2)

i.e. $B \in (-2, 0) \cup (0, \infty)$ (2)

3. (17 points) Consider the following differential equation

$$2\frac{dy}{dx} = (y^2 - 1)\sin x$$

[7 points] (a) Find the general solution of the above differential equation and rewrite it in an explicit form.

[4 points] (b) If $y = \cancel{c}\, k$ is a constant solution of the above differential equation, then find all possible value(s) of $\cancel{c}\, k$

[6 points] (c) Using part (a) and (b), find all singular solutions.

$y \neq \pm 1$ (2)

(a) $\int 2\frac{dy}{y^2-1} = \int \sin x \, dx$

$\int \left(\frac{1}{y-1} \overset{(1)}{-} \frac{1}{y+1}\right) dy = \int \sin x \, dx$

$\overset{(1)}{\ln}\left|\frac{y-1}{y+1}\right| = \underbrace{-\overset{(1)}{\cos x} + \ln c}$

$\frac{y-1}{y+1} = c\, e^{-\cos x} \Rightarrow y = \frac{1 + c\, e^{-\cos x}}{1 - c\, e^{-\cos x}}$ (2)

(b) if $y = k$ solution $\Rightarrow 0 \overset{(1)}{=} (k^2 - 1) \Rightarrow k = \pm 1$ (1)

$\Rightarrow$ (2) $y = 1$ and $y = -1$ on $(-\infty, \infty)$

(c) if $y = 1 \Rightarrow 1 = \frac{1 + Ce^{-\cos x}}{1 - Ce^{-\cos x}}$ (1) $\Rightarrow$ if we set

$C = 0$ (1) we obtain $y = 1 \Rightarrow$ Not a singular solution (1)

if $y = -1 \Rightarrow -1 = \frac{1 + Ce^{-\cos x}}{1 - Ce^{-\cos x}}$ (1)

$-1 + Ce^{-\cos x} = 1 + Ce^{-\cos x} \Rightarrow -1 = 1$ (1) impossible

$y = -1$ is a singular solution (1)

4. (10 points) Find the solution of the IVP

$$\frac{dy}{dx} = -\frac{1}{x}y + \sin x, \quad y(\pi) = 1$$

$x \neq 0 \Rightarrow \quad y' + \frac{1}{x}y = \sin x \quad$ linear 1^{st}-order (2)

(1) $\rightarrow$ $\int \frac{1}{x}dx$ $\leftarrow$ (1)

integrating factor: $\mu = e^{\int \frac{1}{x}dx} = x \qquad x > 0$

(1)

so $\quad xy' + y = x\sin x \Rightarrow \frac{d}{dx}[xy] = x\sin x$

integration by parts gives

$xy = -x\cos x + \sin x + C$ $\leftarrow$ (2)

$\Rightarrow \quad y = -\cos x + \frac{\sin x}{x} + \frac{C}{x}$ (1)

$y(\pi) = 1 \Rightarrow \quad 1 = 1 + \frac{C}{\pi} \Rightarrow C = 0$ $\leftarrow$ (1)

$y = -\cos x + \frac{\sin x}{x}$ (1)

5. (12 points) Solve the differential equation

$$(xe^{2y} - x^2)dx + (x^2e^{2y} + e^y)dy = 0$$

Set $M(x,y) = xe^{2y} - x^2$, $N(x,y) = x^2e^{2y} + e^y$

Then $M_y = \frac{\partial M}{\partial y} = 2xe^{2y}$ 1 ; $N_x = 2xe^{2y}$ 1

So $M_y = N_x$ and then the equation is exact 2

The general solution is $F(x,y) = C$

$\Rightarrow F(x,y) = \int (xe^{2y} - x^2)dx + g(y)$ 1

$= \frac{1}{2}x^2e^{2y} - \frac{x^3}{3} + g(y)$ 2

$\frac{\partial F(x,y)}{\partial y} = N \Rightarrow x^2e^{2y} + g'(y) = x^2e^{2y} + e^y$ 1

so $g'(y) = e^y$ 1 and then

$g(y) = e^y$ 1

The general solution.

$\frac{1}{2}x^2e^{2y} - \frac{x^3}{3} + e^y = C$ 2

6. (10 points) Show that the differential equation

$$(x^3y - y)dx - xdy = 0$$

is not exact and transforms it into an exact equation.

$M(x,y) = x^3y - y$, $N(x,y) = -x$

$M_y = x^3 - 1$ ① $N_x = -1$ ①

So $M_y \neq N_x$ ① and therefore the equation is not exact ①

$\Rightarrow$ Integrating factor

$$\frac{M_y - N_x}{N} = \frac{(x^3-1)-(-1)}{-x} = -x^2 \quad ②$$

depends only on x. An integrating factor would be $\mu(x) = e^{-\int x^2 dx} = e^{-\frac{1}{3}x^3}$ ②

The new exact equation is

$$e^{-\frac{1}{3}x^3}(x^3y - y)\,dx - x e^{-\frac{1}{3}x^3}\,dy = 0 \quad ②$$

7. (12 points)

(a) Solve the homogenouse differential equation: $\dfrac{dy}{dx} = \dfrac{x^2 + y^2}{xy}$

(b) Find an explicit solution of the IVP

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy}, \quad y(e) = 2e$$

ⓐ $y' = \frac{x}{y} + \frac{y}{x} \Rightarrow y' = \left(\frac{y}{x}\right)^{-1} + \frac{y}{x}$ ②

let $v = \frac{y}{x}$ ① $\Rightarrow y = vx \Rightarrow y' = xv' + v$ ①

sub $\Rightarrow xv' + v = \frac{1}{v} + v \Rightarrow xv' = \frac{1}{v} \Rightarrow vv' = \frac{1}{x}$ ①

$\Rightarrow \frac{v^2}{2} = \ln|x| + C$ ②

$\left(\frac{y}{x}\right)^2 = 2\ln|x| + 2C$ ①

ⓑ using $y(e) = 2e$

$\left(\frac{2e}{e}\right)^2 = 2\ln e + 2C \Rightarrow C = 1 \leftarrow$ ②

$\Rightarrow y^2 = 2x^2 \ln|x| + 2x^2$ ①

$y = \pm\sqrt{2x^2 \ln|x| + 2x^2}$ using initial cond.

$y = \sqrt{2x^2 \ln|x| + 2x^2}$ ①

8. (~~13~~ 12 points) Solve the differential equation by using an appropriate substitution

$$3(1+x^2)\frac{dy}{dx} = 2xy(y^3-1)$$

$$y' + \frac{2x}{3(1+x^2)}y = \frac{2x}{3(1+x^2)}y^4 \qquad 2$$

Bernoulli' ⇒ $w = y^{1-4} = y^{-3}$ 1

$$w' = -3y^{-4}y' \qquad 1$$

$$3y^{-4}y' + \frac{2x}{(1+x^2)}y^{-3} = \frac{2x}{(1+x^2)} \qquad 1$$

$$-w' + \frac{2x}{(1+x^2)}w = \frac{2x}{(1+x^2)} \qquad 1$$

$$w' - \frac{2x}{(1+x^2)}w = \frac{-2x}{(1+x^2)} \qquad 1$$

linear 1st-order 1 ⇒ integrating factor $= e^{-\int \frac{2x}{1+x^2}dx}$ 1 $= \frac{1}{1+x^2}$

$$D\left[\frac{1}{1+x^2}w\right] = -2x \qquad 2$$

$$\frac{w}{1+x^2} = \frac{1}{1+x^2} + C$$

$$y^{-3} = 1 + C(1+x^2) \qquad 2$$

9. (10 points) An object is taken from an oven when the temperature is 85^{o}C to a room where the temperature is 25^{o}C. Two minutes later, the temperature of the object is 55^{o}C. How long does it take to reach a temperature of 40^{o}C.

$T_m = 25, \quad T(0) = 85 \quad T(2) = 55$

$$\frac{dT}{T - T_m} = k\,dt \Rightarrow T = T_m + Ae^{kt} = 25 + Ae^{kt} \quad (2)$$

$T(0) = 85 \Rightarrow 85 = 25 + A \Rightarrow A = 60 \quad (2)$

So $T = 25 + 60e^{kt}$

Now $T(2) = 55 \Rightarrow 25 + 60e^{2k} = 55 \quad (1)$

$$60e^{2k} = 30$$

$$k = -\frac{\ln 2}{2} \quad (2)$$

$\Rightarrow T = 25 + 60e^{-\ln 2/2\, t}$

Finally $T(t) = 40 \Rightarrow 25 + 60e^{-\frac{\ln 2}{2}t} = 40 \quad (1)$

$$60e^{-\frac{\ln 2}{2}t} = 15$$

so $e^{-\left(\frac{\ln 2}{2}\right)t} = \frac{1}{4} \Rightarrow t = \underline{4 \text{ minutes}} \quad (1)$