Math201.15, Quizzes # 3 & 4, Term 181

Name: Solutions

ID #: **Serial #:**

1. **[6 points]** Find the local max/min values and saddle points of

$$f(x,y) = 2x^2 + y^4 + 4xy - 2.$$

2. **[6 points]** Find the extreme values of

$$f(x,y,z) = (x-1)^2 + (y-2)^2 + 2(z+1)^2$$

subject to the constraint $x^2 + y^2 + 2z^2 = 7$.

3. **[4 points]** Set up an integral for the volume of the solid that lies under the graph of $z = x^2 + 4y^2 + 8$ and above the region R $(in\ the\ xy-plane)$ bounded by the curves $x + y = 1\ and\ x^2 + y = 1$. **Do not evaluate the integral.**

4. **[4 points]** Evaluate $\int_0^1 \int_{\sqrt{y}}^1 \frac{ye^{3x^2}}{x^3}\, dx\, dy$.

Good luck,

Ibrahim Al-Rasasi

1 $f_x(x,y) = 4x + 4y$; $f_y(x,y) = 4y^3 + 4x$ (both exist at all pts (x,y))

$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} 4x + 4y = 0 \\ 4y^3 + 4x = 0 \end{cases} \Rightarrow \begin{cases} x + y = 0 & \text{—— (1)} \\ y^3 + x = 0 & \text{—— (2)} \end{cases}$ (1)

$(1) \Rightarrow y = -x \xrightarrow{(2)} -x^3 + x = 0 \Rightarrow -x(x^2 - 1) = 0 \Rightarrow -x(x-1)(x+1) = 0$
$\Rightarrow x = 0, 1, -1$ (1)

$x = 0 \Rightarrow y = 0 \Rightarrow (0,0)$
$x = 1 \Rightarrow y = -1 \Rightarrow (1,-1)$ (1.5)
$x = -1 \Rightarrow y = 1 \Rightarrow (-1,1)$

• Test : $f_{xx}(x,y) = 4$, $f_{yy}(x,y) = 12y^2$, $f_{xy}(x,y) = 4$

$D(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - [f_{xy}(x,y)]^2$ (1)
$= 4 \cdot 12y^2 - 16$
$= 16(3y^2 - 1)$

- $(0,0)$: $D(0,0) = -16 < 0 \Rightarrow f$ has a Saddle point at $(0,0)$
  - $f(0,0) = -2$
- $(1,-1)$: $D(1,-1) = 32 > 0$ & $f_{xx}(1,-1) = 4 > 0 \Rightarrow f$ has a local min at $(1,-1)$. The local min value is $f(1,-1) = -3$ 1.5
- $(-1,1)$: $D(-1,1) = 32 > 0$ & $f_{xx}(-1,1) = 4 > 0 \Rightarrow f$ has a local min at $(-1,1)$. The local min value is $f(-1,1) = -3$

$\boxed{2}$ Lagrange Multipliers

$$f(x,y,z) = (x-1)^2 + (y-2)^2 + 2(z+1)^2$$
$$g(x,y,z) = x^2+y^2+2z^2-7$$

- We solve the system

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ g = 0 \end{cases} \Rightarrow \begin{cases} 2(x-1) = \lambda \cdot 2x \\ 2(y-2) = \lambda \cdot 2y \\ 4(z+1) = \lambda \cdot 4z \\ x^2+y^2+2z^2-7=0 \end{cases} \Rightarrow \begin{cases} x-1 = \lambda x & \text{--- (1)} \\ y-2 = \lambda y & \text{--- (2)} \\ z+1 = \lambda z & \text{--- (3)} \\ x^2+y^2+2z^2 = 7 & \text{--- (4)} \end{cases}$$ 1.5

$(\lambda \neq 1$, why?$)$

$(1) \Rightarrow x - \lambda x = 1 \Rightarrow x(1-\lambda) = 1 \Rightarrow x = \frac{1}{1-\lambda} \longrightarrow (5)$

$(2) \Rightarrow y - \lambda y = 2 \Rightarrow y(1-\lambda) = 2 \Rightarrow y = \frac{2}{1-\lambda} \text{ --- } (6)$ 1.5

$(3) \Rightarrow z - \lambda z = -1 \Rightarrow z(1-\lambda) = -1 \Rightarrow z = \frac{-1}{1-\lambda} \text{ --- } (7)$

Substituting 5, 6, 7 in (4):

$$\frac{1}{(1-\lambda)^2} + \frac{4}{(1-\lambda)^2} + \frac{2}{(1-\lambda)^2} = 7 \Rightarrow \frac{7}{(1-\lambda)^2} = 7 \Rightarrow (1-\lambda)^2 = 1 \Rightarrow 1-\lambda = \pm 1$$

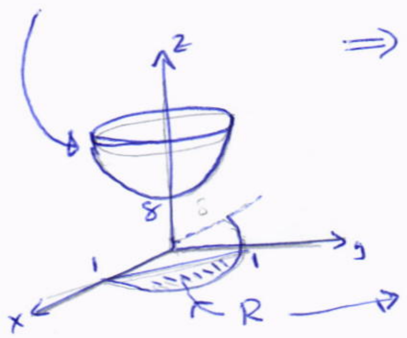
$\Rightarrow \lambda = 0, \ \lambda = 2$ 1

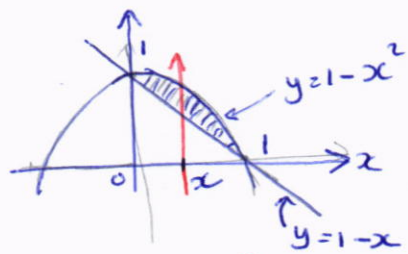
- $\lambda = 0 \xrightarrow{5,6,7} (x,y,z) = (1,2,-1)$ 1
- $\lambda = 2 \xrightarrow{5,6,7} (x,y,z) = (-1,-2,1)$

- $f(1,2,-1) = 0 \leftarrow$ min. value of f 1
- $f(-1,-2,1) = 4 + 16 + 8 = 28 \leftarrow$ max value of f.

$\boxed{3}$ $z = x^2 + 4y^2 + 8$; $x + y = 1$, $x^2 + y = 1$

$\Rightarrow y = 1 - x$, $y = 1 - x^2$





0,5

Type I :

$R: 0 \le x \le 1 , \ 1-x \le y \le 1-x^2$

$$V = \iint_R f(x,y)\, dA \quad 0.5$$

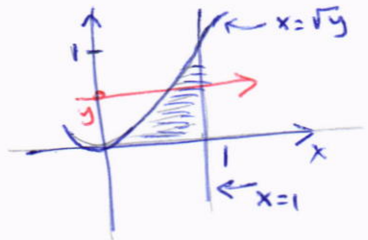
$$= \iint_R (x^2+4y^2+8)\, dA \quad 0.5$$

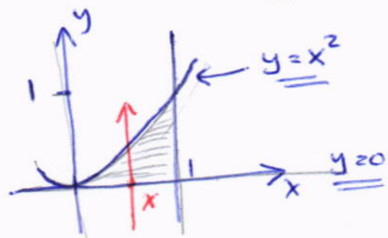
$$= \int_0^1 \int_{1-x}^{1-x^2} (x^2+4y^2+8)\, dy\, dx$$

$\boxed{4}$ $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 \frac{y e^{3x^2}}{x^3}\, dx\, dy$, Can't be integrated $\Rightarrow$ Reverse the order of Integration

Type II $\longrightarrow$ Type I

$R: 0 \le y \le 1 , \ \sqrt{y} \le x \le 1$ (Type II)





$R: 0 \le x \le 1 , \ 0 \le y \le x^2$ (Type I)

$$\int_0^1 \int_{\sqrt{y}}^1 \frac{y e^{3x^2}}{x^3}\, dx\, dy = \int_0^1 \int_0^{x^2} \frac{y e^{3x^2}}{x^3}\, dy\, dx \quad 2$$

$$= \int_0^1 \frac{e^{3x^2}}{x^3} \cdot \frac{1}{2} y^2 \Big]_{y=0}^{y=x^2} dx \quad 0.5$$

$$= \int_0^1 \frac{e^{3x^2}}{x^3} \frac{1}{2} x^4\, dx \quad 0.5$$

$$= \frac{1}{2} \int_0^1 x e^{3x^2}\, dx \quad 0.5$$

$$= \frac{1}{2} \cdot \frac{1}{6} e^{3x^2} \Big]_0^1 \quad 0.5$$

$$= \frac{1}{12} (e^3 - 1) \quad 0.5$$