Math 101

1. If $f(x) = \begin{cases} 5-2x & \text{when} \quad x \le 0 \\ x^2-3x+2 & \text{when} \quad x > 0 \end{cases}$

then $f'_{-}(0) =$
($f'_{-}(0)$: left hand derivative of f at 0)

(a) -2
(b) -3
(c) 5
(d) -1
(e) 0

$$f'_{-}(0) = \lim_{h\to 0^-} \frac{f(h)-f(0)}{h}$$

$$= \lim_{h\to 0^-} \frac{5-2h-5}{h} = \lim_{h\to 0^-} -2 = -2$$

2. If $f(x) = x\sqrt{x} + x \ln x + e + 2^e$, then $f'(x) =$

(a) $\frac{3}{2}\sqrt{x} + \ln x + 1$
(b) $\frac{3}{2}\sqrt{x} + x + \ln x + 2^e \ln 2$
(c) $\frac{3}{2}\sqrt{x} + x \ln x + e$
(d) $\frac{3}{2}\sqrt{x} + x \ln x + 1 + e + 2^e \ln 2$
(e) $\frac{3}{2}\sqrt{x} + x \ln x + 1 + e + e\,2^e \ln 2$

$$f'(x) = \frac{3}{2}\sqrt{x} + x \cdot \frac{1}{x} + \cancel{x}\ln x$$

$$= \frac{3}{2}\sqrt{x} + 1 + \ln x$$

3. If $f(x)$ is differentiable for all $x > 0$ and $f(3x^2+5) = \sqrt[3]{x+7}$, then $f'(8) =$

(a) $\frac{1}{72}$

(b) $\frac{1}{8}$

(c) $\frac{2}{3}$

(d) $\frac{4}{7}$

(e) $\frac{1}{9}$

$$f'(3x^2+5)\cdot 6x = \frac{1}{3}(x+7)^{-2/3}$$

$x = 1$

$$f'(8)\cdot 6 = \frac{1}{3}\, 8^{-2/3}$$

$$f'(8) = \frac{1}{6\cdot 3\cdot 4} = \frac{1}{72}$$

4. Let $f(x) = \dfrac{\sin^{-1} x}{\cos^{-1} x}$, then $f'(0) =$

(a) $\frac{2}{\pi}$

(b) $\frac{\pi}{2}$

(c) does not exist

(d) 1

(e) 0

$$f'(x) = \frac{\dfrac{\cos^{-1}x}{\sqrt{1-x^2}} + \dfrac{\sin^{-1}x}{\sqrt{1-x^2}}}{(\cos^{-1}x)^2}$$

$$f'(0) = \frac{\pi/2 + 0}{(\pi/2)^2} = \frac{2}{\pi}.$$

5. If $f(x) = \dfrac{x}{(1+x)^2}$, then $f'(x) =$

(a) $\dfrac{1-x}{(1+x)^3}$

(b) $\dfrac{1-2x}{(1+x)^4}$

(c) $\dfrac{3+x}{(1+x)^2}$

(d) $\dfrac{1-2x-2x^2}{(1+x)^4}$

(e) $\dfrac{1+x^2}{(1+x)^3}$

$$f'(x) = \frac{(1+x)^2 \cdot 1 - x \cdot 2(1+x)}{(1+x)^4}$$

$$= \frac{(1+x) - 2x}{(1+x)^3}$$

$$= \frac{1-x}{(1+x)^3}$$

6. If the line $y + x = 2$ is tangent to the curve $y = \dfrac{c}{x+2}$, then $c =$

(a) 4
(b) 0
(c) 2
(d) −2
(e) 1

tangent at (α, β), $\alpha \neq -2$

- $-\alpha + 2 = \dfrac{c}{\alpha+2}$ $\quad c = 4 - \alpha^2$

- $-1 = -\dfrac{c}{(\alpha+2)^2}$ $\quad c = (\alpha+2)^2$

$4 - \alpha^2 = (\alpha+2)^2$

$\alpha + 2 = 2 - \alpha$ $\quad \alpha = 0$

$c = (\alpha+2)^2 = 4.$

7. The slope of the tangent line to the curve $y = \dfrac{1 + \sec x}{1 - \tan x}$ at $x = 0$ is

(a) 2

(b) 0

(c) 1

(d) 3

(e) 4

$$y' = \frac{\sec x \tan x (1 - \tan x) + \sec^2 x (1 + \sec x)}{(1 - \tan x)^2}$$

$$y'(0) = \frac{1 \cdot 0 \cdot (1-0) + 1 \cdot (1+1)}{(1-0)^2} = 2$$

8. Let $f(x) = (2x^2 + 3x - 1)\, e^x$, then $\displaystyle\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} =$

(a) $(2x^2 + 7x + 2)\, e^x$

(b) $(4x - 2)\, e^x$

(c) $(2x^2 - x + 3)\, e^x$

(d) $(x + 2h + 1)\, e^{x+h}$

(e) $(x^2 + h^2 + 4)\, e^{x+h}$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

$$f'(x) = (4x+3)e^x + (2x^2+3x-1)e^x$$

$$= (2x^2+7x+2)e^x$$

9. $\lim_{x\to 2} \frac{x^2-4}{\sin(2-x)} =$ $\lim_{x\to 2} \frac{x-2}{\sin(2-\cancel{x})} \cdot (x+2)$

$= \lim_{x\to 2} \frac{-(2-x)}{\sin(2-x)} \cdot \lim_{x\to 2} (x+2$

$= -1 \cdot 4 = -4$

(a) -4

(b) 1

(c) -2

(d) $\frac{-1}{2}$

(e) 0

10. The radius of a cylinder is increasing at a rate of 3 cm/sec and the height is increasing at a rate of 2 cm/sec. How fast is the volume changing when the radius is 1 cm and the height is 4 cm? (Volume of a cylinder is $V = \pi\, r^2\, h$)

(a) $26\,\pi\; cm^3/sec$

(b) $12\,\pi\; cm^3/sec$

(c) $24\,\pi\; cm^3/sec$

(d) $6\,\pi\; cm^3/sec$

(e) $14\,\pi\; cm^3/sec$

$$\frac{dV}{dt} = \pi\, 2\underbrace{r}_{=1}\underbrace{\frac{dr}{dt}}_{=3} \cdot \underbrace{h}_{=4} + \pi \underbrace{r^2}_{=1^2}\underbrace{\frac{dh}{dt}}_{=2}$$

$= 26\pi$

11. If $x = \cot 2y$, then $\dfrac{dy}{dx} =$

(a) $-\dfrac{1}{2}\sin^2 2y$

(b) $-y\sin^2 2y$

(c) $\cos^2 2y$

(d) $-\dfrac{1}{2}\cos^2 2y$

(e) $-x\sin^2 2y$

$1 = -\operatorname{cosec}^2 2y \cdot 2y'$

$1 = -\dfrac{1}{\sin^2 2y} 2y'$

$y' = -\dfrac{1}{2}\sin^2 2y$

12. Which one of the following statements is **TRUE**?

(a) If f is differentiable, then $\dfrac{d}{dx}\sqrt{f(x)} = \dfrac{f'(x)}{2\sqrt{f(x)}}$

(b) If f is differentiable, then $\dfrac{d}{dx} f\left(\sqrt{x}\right) = \dfrac{f'(x)}{2\sqrt{x}}$

(c) $\dfrac{d}{dx}(10^x) = x\,10^{x-1}$

(d) $\dfrac{d}{dx}(\ln 10) = \dfrac{1}{10}$

(e) $\dfrac{d}{dx}|x^2 + x| = |2x + 1|$

13. The position of a particle at time t is given by the equation

$$s(t) = 2t^3 - 15t^2 + 36t.$$

What is the total distance traveled by the particle during $0 \le t \le 3$?

(a) 29
(b) 17
(c) 11
(d) 9
(e) 28

$v(t) = s'(t) = 6t^2 - 30t + 36$

$= 6(t-2)(t-3)$

$v(t) > 0, \quad \alpha t < 2$

$v(t) < 0, \quad 2 < t < 3$

$) = |s(2) - s(0)| + |s(3) - s(2)| = 28 + 1 = 29$

14. If $y = (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)$, $x > 0$, then $\frac{dy}{dx}\big|_{x=1} =$

(a) 240
(b) 320
(c) 120
(d) 75
(e) 400

$\ln y = \ln(1+x) + \ln(1+x^2) + \ln(1+x^3) + \ln(1+x^4)$
$+ \ln(1+x^5$

$\frac{y'}{y} = \frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{3x^2}{1+x^3} + \frac{4x^3}{1+x^4} + \frac{5x^4}{1+x^5}$

$x = 1$

$y' = 2^5\left(\frac{1}{2} + \frac{2}{2} + \frac{3}{2} + \frac{4}{2} + \frac{5}{2}\right)$

$= 240$

15. If $y = \dfrac{1}{(2x+3)}$, then $y^{(63)}\left(\dfrac{-1}{2}\right) =$

(a) $-\dfrac{63!}{2}$

(b) $\dfrac{63}{2^{64}}$

(c) $63!$

(d) $-\dfrac{63}{2^{63}}$

(e) $\dfrac{1}{2^{64}}$

$$y' = \frac{(-1)2}{(2x+3)^2}$$

$$y'' = \frac{(-1)(-2)2^2}{(2x+3)^3}$$

$$y^{(3)} = \frac{(-1)(-2)(-3)2^3}{(2x+3)^4}$$

$$\vdots$$

$$y^{(n)} = \frac{(-1)(-2)\ldots(-n)2^n}{(2x+3)^{n+1}} = \frac{(-1)^n n!\,2^n}{(2x+3)^{n+1}}$$

$$y^{(63)}\left(-\tfrac{1}{2}\right) = \frac{-63!\,2^{63}}{2^{64}} = -\frac{63!}{2}$$

16. If $u = e^{y^3+2y}$ and $x = (u-1)\,e^u$ then $\dfrac{dy}{dx}\Big|_{x=0} =$

(a) $\dfrac{1}{2e}$

(b) 0

(c) $\dfrac{1}{2}e$

(d) 1

(e) 2

$x=0 \quad 0 = (u-1)e^u \rightarrow u=1$

$u=1 \quad 1 = e^{y^3+2y}$

$y(y^2+2)=0 \rightarrow y=0$

$1 = e^{y^3+2y}(3y^2+2)\dfrac{dy}{du} \Rightarrow \dfrac{dy}{du}\Big|_{y=0} = \dfrac{1}{2}$

$1 = ((u-1)e^u + e^u)\dfrac{du}{dx} \quad u=1 \rightarrow \dfrac{du}{dx} = \dfrac{1}{e}$

$$\frac{dy}{dx} = \frac{dy}{dx}\,\frac{du}{dx} = \frac{1}{2}\cdot\frac{1}{e} = \frac{1}{2e}$$

17. $\lim_{\theta \to 0} \frac{1 - \cos\theta}{(\theta \tan\theta)(1 + \sec\theta)} =$

(a) $\frac{1}{4}$

(b) $\frac{1}{2}$

(c) 0

(d) $\frac{-1}{2}$

(e) $\frac{-1}{4}$

$$\lim_{\theta \to 0} \frac{(1-\cos\theta)(1+\cos\theta)}{\theta \frac{\sin\theta}{\cos\theta}(1+\sec\theta)(1+\cos\theta)}$$

$$= \lim_{\theta \to 0} \frac{\cos\theta \sin^2\theta}{\theta \cancel{\sin\theta}((1+\sec\theta)(1+\cos\theta)}$$

$$= \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \frac{\cos\theta}{(1+\sec\theta)(1+\cos\theta)} = 1 \cdot \frac{1}{4}$$

$$= 1/4$$

18. If $F(x) = f(x\,g(\sin x))$ where $f'(\pi) = 1, g'(0) = \frac{2}{\pi}$, $g(0) = 1$ and $g(\pi) = 6$, then $F'(\pi) =$

(a) -1

(b) -2

(c) 1

(d) 2

(e) 0

$$F'(x) = f'(x\,g(\sin x))\,(g(\sin x) + x g'(\sin x)\cos x$$

$$F'(\pi) = f'(\pi\,g(0))\,(g(0) + \pi g'(0)(-1))$$

$$= f'(\pi)\left(1 - \pi \frac{2}{\pi}\right) = -1$$

$y_1 = x^{\cos x}$ $y_2 = x^{-1}$

19. If $y = x^{\cos x} + x^{-1}$, then $y'\left(\frac{\pi}{2}\right) =$

$\ln y_1 = \cos x \ln x$

(a) $\ln 2 - \ln \pi - \frac{4}{\pi^2}$

(b) $\ln 2 + \ln \pi$

(c) 0

(d) $\ln 2 + \ln \pi + \frac{4}{\pi^2}$

(e) $\pi + 2$

$\frac{y_1'}{y_1} = \frac{\cos x}{x} - \sin x \ln x$

$y_1' = x^{\cos x}\left(\frac{\cos x}{x} - \sin x \ln x\right)$

$y_1'(\pi/2) = \pi/2^{0}\left(\frac{0}{\pi/2} - 1 \cdot \ln \pi/2\right)$

$= \ln 2 - \ln \pi$

$y_2' = -x^{-2}$ $y_2'(\pi/2) = -4/\pi^2$ $y' = \ln 2 - \ln \pi - 4/\pi^2$

20. A particle is moving along the circle $x^2 + y^2 = 25$. As it passes through the point $(3,4)$, its x-coordinate is changing at a rate of $2\,cm/sec$. How fast is the y-coordinate changing at that instant?

(a) $\frac{-3}{2}\,cm/sec$

(b) $\frac{-1}{2}\,cm/sec$

(c) $-1\,cm/sec$

(d) $1\,cm/sec$

(e) $2\,cm/sec$

$x^2 + y^2 = 25$

$\underset{3}{\underset{\shortparallel}{2x}}\ \underset{2}{\underset{\shortparallel}{\frac{dx}{dt}}} + \underset{4}{\underset{\shortparallel}{2y}}\ \frac{dy}{dt} = 0$

$\frac{dy}{dt} = -\frac{12}{8} = -3/2$