King Fahd University of Petroleum and Minerals

Department of Mathematics and Statistics
Math 333 Major 1 2017–2018 (173)

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Name:
ID #
Section #

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Prob. 1: Find the directional derivative of $f(x, y) = y \cos xz$ at $(1, -\sqrt{\pi}, 0)$ in the direction of $v =< 1, 2, 3 >$.

3 We have $\frac{\partial f}{\partial x} = -yz \sin xz$; $\frac{\partial f}{\partial y} = \cos xz$; $\frac{\partial f}{\partial z} = -xy \sin xz$

3 and $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{<1,2,3>}{\|\vec{v}\|} = <\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}>$

Therefore

$$D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u} = \left.\begin{pmatrix} -yz \sin xz \\ \cos xz \\ -xy \sin xz \end{pmatrix}\right|_{(1,-\sqrt{\pi},0)} \cdot \begin{pmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{pmatrix} = 2/\sqrt{14}$$

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Prob. 2: Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ be a constant vector and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Verify the identity $\nabla \times [(\mathbf{r}.\mathbf{r})\mathbf{a}] = 2(\mathbf{r} \times \mathbf{a})$.

The LHS is equal to $\nabla \times [(x^2+y^2+z^2)\vec{a}]$ 2

2 $= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (x^2+y^2+z^2)a_1 & (\dots)a_2 & (\dots)a_3 \end{vmatrix} = \langle 2ya_3 - 2za_2, 2za_1 - 2xa_2, 2xa_2 - 2ya_1 \rangle$

$= 2\langle ya_3 - za_2, za_1 - xa_2, xa_2 - ya_1 \rangle$

3

2 $= 2 \begin{vmatrix} i & j & k \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix} = 2(\vec{r} \times \vec{a}) = RHS$

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Prob. 3: Evaluate $\int_C \frac{z^2}{x^2+y^2} ds$ where C is given by $x = \cos 2t$, $y = \sin 2t$, $z = 2t$, $\pi \leq t \leq 2\pi$.

Clearly, the integral is equal to

$$\int_{\pi}^{2\pi} \frac{4t^2}{\cos^2 2t + \sin^2 2t} \sqrt{(-2\sin 2t)^2 + (2\cos 2t)^2}\, dt$$ 4

$$= \int_{\pi}^{2\pi} 4t^2\, 2\sqrt{2}\, dt = \frac{8\sqrt{2}}{3} t^3 \Big|_{\pi}^{2\pi} = \frac{56\sqrt{2}}{3}\pi^3$$

2 2

16 **Prob. 4**: An object is moved by a force $\mathbf{F}(x,y,z) = 2xz\mathbf{i} + 2yz\mathbf{j} + (x^2 + y^2 + 1)\mathbf{k}$ through the curve $C = \{r(t) = (e^{t^2}, t, \sin t),\ 0 \le t \le \pi\}$. Use the fundamental theorem of calculus to find the work done by the force $\mathbf{F}$.

First, we prove that it is path independent

$$\operatorname{Curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xz & 2yz & x^2+y^2+1 \end{vmatrix} = (2y-2y)\vec{i} - (2x-2x)\vec{j} + (0-0)\vec{k} = \vec{0} \quad 3$$

Therefore $\vec{F}$ is conservative since it has continuous partial derivatives and $\operatorname{Curl}\vec{F} = \vec{0}$.

Let us find the potential ϕ so that $\vec{F} = \vec{\nabla}\phi$. 1

We have $\phi_x = P = 2xz \Rightarrow \phi = x^2 z + g(y,z)$. 2

Next, $\phi_y = g'_y(y,z) = 2yz \Rightarrow g(y,z) = y^2 z + h(z)$

$\Rightarrow \phi = x^2 z + y^2 z + h(z)$. 2

We also have $\phi_z = x^2 + y^2 + h'(z) = x^2 + y^2 + 1$. 2

We deduce that $h'(z) = 1 \Rightarrow h(z) = z + C$. 1

Hence, $\phi(x,y,z) = x^2 z + y^2 z + z + C$. 1

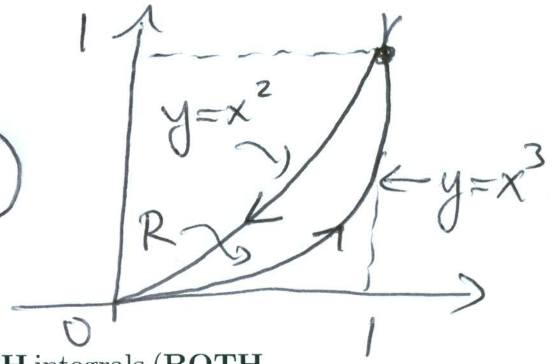
Using the FTC, we end up with

$$W = \int_C \vec{F} \cdot d\vec{r} = \phi(B) - \phi(A) = \phi(r(\pi)) - \phi(r(0)) \quad 2$$

$$= \phi(e^{\pi^2}, \pi, 0) - \phi(1,0,0) = 0 - 0 = 0 \quad 2$$

Green's Thm:

$$\oint_C P\,dx+Q\,dy=\iint_R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA$$ 2



14 **Prob. 5**: Verify Green's theorem by evaluating **BOTH** integrals (**BOTH** sides of the identity in Green's theorem) with $\mathbf{F}(x,y) =< x^2-y^2, 2y-x >$ and C consists of the boundary of the region in the first quadrant that is bounded by the graph of $y=x^2$ and $y=x^3$.

As we have a curve C which is smooth and bounding a simply connected region R and $\frac{\partial Q}{\partial x}=-1$, $\frac{\partial P}{\partial y}=-2y$ are continuous on R, we can apply Green's theorem. 1

The LHS of the identity is equal to

$$\int_C P\,dx+Q\,dy=\int_{y=x^3}+\int_{y=x^2}=\int_0^1(x^2-x^6)\,dx+(2x^3-x)\,3x^2\,dx$$ 2

$$+\int_1^0(x^2-x^4)\,dx+(2x^2-x)\,2x\,dx=-\frac{11}{420}$$ 1

2

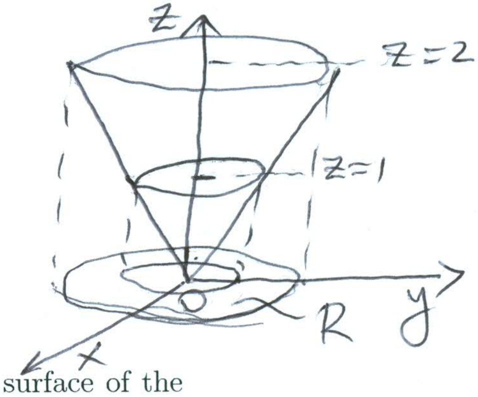
The RHS is equal to

2 $$\iint_R(-1+2y)\,dA=\int_0^1\int_{x^3}^{x^2}(-1+2y)\,dy\,dx=\int_0^1\left[-y+y^2\right]_{x^3}^{x^2}dx$$

2 1

$$=-\frac{11}{420}$$ 1

The theorem is therefore verified.



14 **Prob. 6**: Evaluate the integral $\int\int_S z^2 ds$ where S is the surface of the cone $z^2 = x^2 + y^2$ between the planes $z = 1$ and $z = 2$.

Let R be the projection of the surface S on the xy-plane

$R = \{(x,y) \in \mathbb{R}^2 : 1 \leq x^2+y^2 \leq 2\}$. 1

We have

$$\iint_S z^2 dS = \iint_R (x^2+y^2)\sqrt{1+\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2+\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2}\, dx\,dy \quad 6$$

$$= \iint_R (x^2+y^2)\sqrt{2}\, dx\,dy = \int_1^2\int_0^{2\pi} \sqrt{2}\, r^2 . r\,dr\,d\theta = 2\sqrt{2}\,\pi \int_1^2 r^3 dr$$

1 2 1

$$= \frac{15\sqrt{2}}{2}\pi \quad 1$$

14 **Prob. 7**: Use Stoke's theorem to evaluate

$$\oint_C -2ydx + 3xdy + 10zdz$$

where C is the circle $(x-1)^2 + (y-3)^2 = 25$, $z = 3$.

As C is a smooth closed simple curve and we can consider the disc enclosed by this curve which is a smooth orientable surface and P, Q, R are continuous with continuous partial derivatives, we can apply Stoke's thm.

2 $\oint_C \vec{F}.d\vec{r} = \iint_S \text{curl}\vec{F}.\vec{n}\,dS$ where $\text{curl}\vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \frac{\partial}{\partial z} \\ -2y & 3x & 10z \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}$

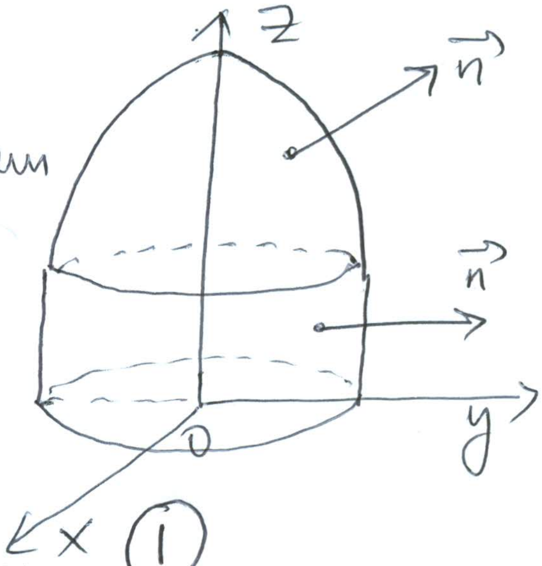
and $\vec{n} = \vec{k}$ 3 3

Therefore, $\ldots = \iint_R 5\vec{k}.\vec{k}\,dA = 5 \times (\text{Area of } R) = 125\pi$

3 2 1

(14)

Prob. 8: Use the divergence theorem to evaluate $\int\int_S (\mathbf{F}.\mathbf{n})dS$ where $\mathbf{F} = xy\mathbf{i} - \frac{1}{2}y^2\mathbf{j} + z\mathbf{k}$ and the surface consists of three parts: $z = 4 - 3x^2 - 3y^2$, $1 \le z \le 4$ on the top, $x^2 + y^2 = 1$, $0 \le z \le 1$ on the sides and $z = 0$ on the bottom.



Take $x = r\cos\theta$, $y = r\sin\theta$

We can apply the divergence theorem because the region is closed and bounded and the surface is smooth and can be oriented outward. In addition P, Q, R are continuous and have continuous partial derivatives. (1)

We have the description $0 \le z \le 4 - 3r^2$ (2) ; $0 \le r \le 1$ (1) and $0 \le \theta \le 2\pi$. (1)

Moreover, $\operatorname{div}\vec{F} = y - y + 1 = 1$. (2)

Therefore,

(2) $$\iint_S (\vec{F}\cdot\vec{n})dS = \iiint_V \operatorname{div}\vec{F}\,dV = \int_0^{2\pi}\int_0^1\int_0^{4-3r^2} r\,dz\,dr\,d\theta \quad (1)$$

$$= \int_0^{2\pi}\int_0^1 (4r - 3r^3)\,dr\,d\theta = \int_0^{2\pi}\left(2r^2 - \frac{3r^4}{4}\right)\Big|_0^1 d\theta = \frac{5\pi}{2}.$$

(2) (1) (1)