01

King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 302 Major Exam I
The Third Semester of 2017-2018 (173)

Time Allowed: 120 Minutes

Name: Key ID#:

Section/Instructor: Serial #:

- Mobiles and calculators are not allowed in this exam.
- Write neatly and eligibly. You may lose points for messy work.
- Show all your work. No points for answers without justification.

Question #	Marks	Maximum Marks
1		25
2		10
3		06
4		20
5		20
6		19
Total		100

Q:1 (5 + 8 + 8 + 4 = 25 points) (a) Determine whether $E = \{< a, b, c, s >: abc = s$ and $a, b, c, s \in R\}$ is a subspace of R^4 ?

Sol: Notice that $0.0.0 = 0 \Rightarrow 0 \in E$ 1

Let $\vec{u} = \langle 1,1,1,1 \rangle \in E$ and $\vec{v} = \langle 1,2,3,6 \rangle \in E$ 2

$\vec{u} + \vec{v} = \langle 2,3,4,7 \rangle \notin E$

$2 \cdot 3 \cdot 4 \neq 7$ 2

$\Rightarrow$ E is not a subspace of R^4.

(b) Show that $E = \{< x, y, z, w >: x + 2y + z = w$ and $x, y, z, w \in R\}$ is a subspace of R^4.

(i) E is nonempty since the vector $\langle 0,0,0,0 \rangle$ is in the set. 1

(ii) Let $\vec{u} = \langle x_1, y_1, z_1, w_1 \rangle$, $\vec{v} = \langle x_2, y_2, z_2, w_2 \rangle$ be in E. 2

Then $x_1 + 2y_1 + z_1 = w_1$ and $x_2 + 2y_2 + z_2 = w_2$,

We have
$\vec{u} + \vec{v} = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2, w_1 + w_2 \rangle$ 2

and $(x_1 + x_2) + 2(y_1 + y_2) + (z_1 + z_2)$

$= (x_1 + 2y_1 + z_1) + (x_2 + 2y_2 + z_2)$ 1

$= w_1 + w_2$

$\Rightarrow \vec{u} + \vec{v}$ is also in E

(ii) Let k be a scalar, Then $k\vec{u} = \langle kx_1, ky_1, kz_1, kw_1 \rangle$

We have $kx_1 + k2y_1 + kz_1 = k(x_1 + 2y_1 + z_1)$

$= k\vec{w}_1$ 2

$\Rightarrow k\vec{u}$ is also in E.

Hence E is a subspace of R^4,

(c) Find a basis and dimension of $E = \{(x, y, z, w) : x + 2y + z = w \text{ and } x, y, z, w \in R\}$.

Let $\vec{u} = \langle x, y, z, w \rangle \in E$

Since $x + 2y + z = w$, we have

$\vec{u} = \langle x, y, z, w \rangle = \langle x, y, z, x + 2y + z \rangle$ 1

$= \langle x, 0, 0, x \rangle + \langle 0, y, 0, 2 \rangle + \langle 0, 0, z, z \rangle$

$= x \langle 1, 0, 0, 1 \rangle + y \langle 0, 1, 0, 2 \rangle + z \langle 0, 0, 1, 1 \rangle$ 3

$B = \{\langle 1, 0, 0, 1 \rangle, \langle 0, 1, 0, 2 \rangle, \langle 0, 0, 1, 1 \rangle\}$ span E.

Since matrix $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ has rank 3, 2

$\Rightarrow$ Set B is L.I. Thus B is a basis for E, 1

$\dim E = 3$. 1

(d) Determine whether the vectors $\mathbf{u_1} = \langle 2, 4, 6, 8 \rangle$, $\mathbf{u_2} = \langle 1, 1, 1, 1 \rangle$, $\mathbf{u_3} = \langle 8, 6, 4, 2 \rangle$ are linearly independent. Give reason for your answer.

$$M = \begin{pmatrix} 2 & 4 & 6 & 8 \\ 1 & 1 & 1 & 1 \\ 8 & 6 & 4 & 2 \end{pmatrix} \underset{\sim}{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 6 & 8 \\ 8 & 6 & 4 & 2 \end{pmatrix}$$ 2

$$\underset{R_3 - 8R_1}{\overset{R_2 - 2R_1}{\sim}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 6 \\ 0 & -2 & -4 & -6 \end{pmatrix} \underset{\sim}{R_3 + R_2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since rank (A) = 2 but the number of vectors is 3, the given set is linearly dependent. 2

Q:2 (10 points) Solve the following nonhomogeneous system using **Gaussian Elimination method:**

$$2x_1 - 5x_2 + 7x_3 = 19$$

$$4x_1 + x_2 + 3x_3 = 5$$

$$x_1 + 3x_2 - 2x_3 = -7$$

$$(A \vdots B) = \left(\begin{array}{ccc|c} 2 & -5 & 7 & 19 \\ 4 & 1 & 3 & 5 \\ 1 & 3 & -2 & -7 \end{array}\right) \underset{\sim}{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 3 & -2 & -7 \\ 4 & 1 & 3 & 5 \\ 2 & -5 & 7 & 19 \end{array}\right) \quad 1$$

$$2 \quad \underset{\sim}{\begin{array}{c} R_2 - 4R_1 \\ R_3 - 2R_1 \end{array}} \left(\begin{array}{ccc|c} 1 & 3 & -2 & -7 \\ 0 & -11 & 11 & 33 \\ 0 & -11 & 11 & 33 \end{array}\right) \underset{\sim}{\begin{array}{c} -\frac{R_2}{11} \\ -\frac{R_3}{11} \end{array}} \left(\begin{array}{ccc|c} 1 & 3 & -2 & -7 \\ 0 & 1 & -1 & -3 \\ 0 & 1 & -1 & -3 \end{array}\right) \quad 2$$

$$\underset{\sim}{R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 3 & -2 & -7 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad 1$$

$$\Rightarrow \begin{array}{l} x_1 + 3x_2 - 2x_3 = -7 \\ \quad\quad x_2 - x_3 = -3 \end{array} \quad 1$$

Let $x_3 = t$. Then $x_2 = t - 3$ and $x_1 = -7 + 2t - t + 3$
$= t - 4$ 3

$\Rightarrow$ System has infinitely many solutions.

Q:3 (04 + 02 = 06 points) Let A be a non-zero 11×13 matrix .

(a) If $\text{rank}(A|B) = 9$, then for what value(s) of rank A is the system $AX = B$, $B \neq 0$ inconsistent? consistent?

The system is inconsistent if rank (A) < 9 2

The system is consistent if rank (A) = rank (A|B) = 9 2

(b) If rank $(A) = 8$, then how many parameters (free variables after performing Gaussian Elimination) does the solution of the system $AX = 0$ have?

$n = 13, \quad r(A) = 8$

Solution of the system has $n - r = 13 - 8 = 5$ parameters.

2

Q:4 (16 points) (a) Use Gauss-Jordan elimination method to find matrix A if $A^{-1} = \begin{pmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{pmatrix}$.

We compute $(A^{-1})^{-1} = A$.

$$(A^{-1} : I) = \left(\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ -2 & 3 & 4 & 0 & 1 & 0 \\ -5 & 5 & 6 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\frac{R_1}{2}} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -2 & 3 & 4 & 0 & 1 & 0 \\ -5 & 5 & 6 & 0 & 0 & 1 \end{array}\right) \quad 1$$

$$\xrightarrow[R_3+5R_1]{R_2+2R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 5 & \frac{17}{2} & \frac{5}{2} & 0 & 1 \end{array}\right) \xrightarrow[\frac{R_3}{5}]{\frac{R_2}{3}} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{5}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{17}{10} & \frac{1}{2} & 0 & \frac{1}{5} \end{array}\right) \quad 4 + 3$$

$$\xrightarrow{R_3-R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{5}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{30} & \frac{1}{6} & -\frac{1}{3} & +\frac{1}{5} \end{array}\right) \quad 2$$

$$\xrightarrow{30R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{5}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 5 & -10 & +6 \end{array}\right) \quad 2$$

$$\xrightarrow[R_2-\frac{5}{3}R_3]{R_1-\frac{1}{2}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 5 & -3 \\ 0 & 1 & 0 & -8 & 17 & -10 \\ 0 & 0 & 1 & 5 & -10 & 6 \end{array}\right) \quad 4$$

(b) (04 points) Solve the system $\mathbf{AX} = \mathbf{B}$, where A is the matrix found in (a), $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$.

$$X = A^{-1} B = \begin{pmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \quad 1$$

$$= \begin{pmatrix} 4 \\ 6 \\ 7 \end{pmatrix} \quad 3$$

Q:5 (10 + 10 = 20 points) (a) Find values of $a \neq 0$ and $b \neq 0$ so that the nonsingular matrix

$$A = \begin{pmatrix} \frac{1}{2} & a \\ b & \frac{1}{2} \end{pmatrix} \text{ is orthogonal.}$$

$|A| = \frac{1}{4} - ab \neq 0 \Rightarrow A$ is nonsingular matrix.

We know that $A A^T = I$

$$\begin{pmatrix} \frac{1}{2} & a \\ b & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & b \\ a & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

4

$$\begin{pmatrix} \frac{1}{4} + a^2 & \frac{a+b}{2} \\ \frac{a+b}{2} & \frac{1}{4} + b^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2

$$\Rightarrow \begin{aligned} \frac{1}{4} + a^2 &= 1 \\ \frac{1}{4} + b^2 &= 1 \\ a+b &= 0 \end{aligned} \quad \Rightarrow a = \pm \frac{\sqrt{3}}{2}, \quad b = \pm \frac{\sqrt{3}}{2}$$

But $a+b=0$ indicates a and b must have opposite sign.

Therefore, $a = -\frac{\sqrt{3}}{2}$, $b = \frac{\sqrt{3}}{2}$ or $a = \frac{\sqrt{3}}{2}$, $b = -\frac{\sqrt{3}}{2}$.

4

(b) Let $A = \begin{pmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & -8 \end{pmatrix}$.

The matrix A has $\lambda_1 = \lambda_2 = -9$ and $\lambda_3 = 9$ as eigenvalues. For λ_1, we have 2 corresponding eigenvectors

$K_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $K_2 = \begin{pmatrix} 1 \\ -4 \\ 0 \end{pmatrix}$, and for λ_3 we have $K_3 = \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}$.

(i) Is $\{K_1, K_2, K_3\}$ an orthogonal set ? If not, find an orthogonal set of eigenvectors.

(ii) Find an orthonormal set of eigenvectors.

Hint: Eigenvectors corresponding to λ_1 have the form $\begin{pmatrix} \frac{\beta}{4} - \frac{\alpha}{4} \\ \alpha \\ \beta \end{pmatrix}$.

Sol.

(b) $K_1^T K_2 \neq 0$

$K_1^T K_3 = 0$

$K_2^T K_3 = 0$

3

Cont.

(3) $\{K_1, K_2, K_3\}$ is not an orthogonal set because K_1 and K_2 are not orthogonal.
We will replace K_2 by K_2^* which is still an eigenvector but is orthogonal to K_1.

Now $\begin{pmatrix} \frac{\beta}{4} - \frac{\alpha}{4} \\ \alpha \\ \beta \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} = 0$ (2)

$\Rightarrow \quad \alpha + \beta = 0 \quad \Rightarrow \quad \alpha = -\beta$

Therefore $K_2^* = \begin{pmatrix} \beta/2 \\ -\beta \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ or $\begin{pmatrix} \frac{1}{2} \\ -1 \\ 1 \end{pmatrix}$ (2)

(ii) Normalizing the eigenvectors

$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{pmatrix}, \quad \begin{pmatrix} \frac{4}{3\sqrt{2}} \\ \frac{1}{3\sqrt{2}} \\ \frac{1}{3\sqrt{2}} \end{pmatrix}$ (3)

$\|K_1\| = \sqrt{2}, \quad \|K_2\| = 3, \quad \|K_3\| = \sqrt{18} = 3\sqrt{2}$

Q:6 (7 + 12 = 19 points) (a) Is $A = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$ diagonalizable ? (Justify your answer).

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 1 \\ 6 & -1-\lambda & 0 \\ -1 & -2 & -1-\lambda \end{vmatrix} = 0 \qquad 2$$

$$\Rightarrow (1-\lambda)\left[\lambda^2+1+2\lambda\right] - 2\left[-6-6\lambda\right] + 1\left[-12-1-\lambda\right] = 0$$

$$\Rightarrow \lambda^2+1+2\lambda-\lambda^3-\lambda-2\lambda^2+12+12\lambda-13-\lambda = 0 \qquad 2$$

$$\Rightarrow -\lambda^3 - \lambda^2 + 12\lambda = 0$$

$$\Rightarrow \lambda(\lambda^2+\lambda-12) = 0 \quad \Rightarrow \lambda(\lambda+4)(\lambda-3) = 0$$

$$\Rightarrow \lambda = 0, -4, 3 \qquad 3$$

Since the eigenvalues are distinct, A is diagonalizable.

(b) Let $A = \begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix}$.

(i) Explain why A is orthogonally diagonalizable.

(ii) Find an ortogonal matrix P that diagonalizes A and find the diagonal matrix $D = P^T AP$.

(i) Since A is symmetric matrix.

2

(ii) $|A - \lambda I| = \begin{vmatrix} 3-\lambda & 2 \\ 2 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda(\lambda-3) - 4 = 0$

$$\Rightarrow \lambda^2 - 3\lambda - 4 = 0$$

$$\Rightarrow (\lambda-4)(\lambda+1) = 0$$

$$\Rightarrow \lambda = 4, -1 \qquad 2$$

For $\lambda_1 = -1$: $K_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

For $\lambda_2 = 4$: $K_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ 2

K_1 and K_2 are orthogonal eigenvectors $(K_1^T K_2 = 0)$

Orthogonal matrix $P = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$ 3

$$D = P^T A P = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \qquad 2$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix} \qquad 1$$