1. Solve the IVP: $y'' = \dfrac{1}{(x+1)^2}$, $y(0) = 2$, $y'(0) = 1$.

Solution.
$y' = \int \dfrac{1}{(x+1)^2}dx + C_1 = -\dfrac{1}{x+1} + C_1$, so $y = -\int \dfrac{1}{x+1}dx + C_1 x + C_2 = -\ln|x+1| + C_1 x + C_2$.
$y(0) = 2$ gives $C_2 = 2$ and $y'(0) = 1$ gives $C_1 = 2$.
IVP solution is $y = -\ln|x+1| + 2x + 2$.

2. Find a general solution of the DE: $(3y^2 - y)\, x^2 \dfrac{dy}{dx} = (x-1)\, y^4$.

Solution. DE is $\dfrac{3y^2 - y}{y^4}dy = \dfrac{x-1}{x^2}dx$ and so it is separable.
Integration gives the family of solutions $-\dfrac{3}{y} + \dfrac{1}{2y^2} = \ln|x| + \dfrac{1}{x} + C$ (i.e. $x - 6xy = 2y^2 (Cx + x\ln|x| + 1)$.

3. Find a general solution of the DE: $(\cos x)\, y' + (\sin x)\, y = 1 \quad (-\pi/2 < x < \pi/2)$.
Solution. Standard form of DE is $y' + (\tan x)\, y = \sec x$.
Integrating factor is $y = e^{\int \tan x dx} = \sec x$.
Hence $y \sec x = \int \sec^2 x dx + C$.
This gives the family of solutions $y = \cos x\,(\tan x + C)$ i.e. $y = \sin x + C\cos x$.

4. Verify that the following DE is exact; then solve it: $(e^y + y\cos x)\, dx + \left(xe^y + \sin x + \dfrac{1}{1+y^2} \right) dy = 0$.

Solution. Let $M = e^y + y\cos x$ and $N = xe^y + \sin x + \dfrac{1}{1+y^2}$.
Then $M_y = e^y + \cos x = N_x$, hence the DE is exact.
This means there is a function F (of x and y) such that $F_x = M$ and $F_y = N$.
Hence $F = xe^y + y\sin x + g(y)$.
From $F_y = N$, we get $xe^y + \sin x + g'(y) = xe^y + \sin x + \dfrac{1}{1+y^2}$.
So $g'(y) = \dfrac{1}{1+y^2}$, i.e. $g(y) = \tan^{-1} y$.
DE has family of solutions: $xe^y + y\sin x + \tan^{-1} y = C$.

5. <u>Use the adjoint matrix</u> to find the inverse of $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & -1 \\ 0 & 1 & 1 \end{bmatrix}$.

Solution. The cofactor matrix of A is $\begin{bmatrix} 4 & -2 & 2 \\ -1 & 1 & -1 \\ 3 & -1 & 3 \end{bmatrix}$.

The adjoint of A is $\begin{bmatrix} 4 & -2 & 2 \\ -1 & 1 & -1 \\ 3 & -1 & 3 \end{bmatrix}^T$, i.e. $adj A = \begin{bmatrix} 4 & -1 & 3 \\ -2 & 1 & -1 \\ 2 & -1 & 3 \end{bmatrix}$.

Determinant of A is 2. We have $A^{-1} = \dfrac{1}{\det A} adj A$, hence $A^{-1} = \dfrac{1}{2}\begin{bmatrix} 4 & -1 & 3 \\ -2 & 1 & -1 \\ 2 & -1 & 3 \end{bmatrix}$.

6. Determine whether the vectors $u = (3, 2, 5)$, $v = (1, -1, 3)$, $w = (1, 4, -1)$ are linearly dependent or independent. If they are linearly dependent, write one of the vectors as a linear combination of the other two vectors.

Solution.

$$\begin{bmatrix} 3 & 1 & 1 \\ 2 & -1 & 4 \\ 5 & 3 & -1 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 2 & -3 \\ 2 & -1 & 4 \\ 5 & 3 & -1 \end{bmatrix} \xrightarrow[R_3 - 5R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & -3 \\ 0 & -5 & 10 \\ 0 & -7 & 14 \end{bmatrix} \xrightarrow{(-1/5)R_2} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & -7 & 14 \end{bmatrix} \xrightarrow{R_3 + 7R_2} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

The last matrix above is an echelon form of the matrix $[u \vdots v \vdots w]$ and contains a zero row, so the vectors u, v, w are linearly dependent.

Let $xu + yv + zw = \mathbf{0}$, then (using the echelon matrix obtained) $y = 2z$ and $x = -2y + 3z = -z$.

Putting $z = 1$ we get $x = -1$, $y = 2$.

Hence $-u + 2v + w = 0$, i.e. $w = u - 2v$ (or, if we want, $u = 2v + w$).

7. Express the vector $v = (0, -2, 7)$ as a linear combination of the vectors

$u_1 = (5, 1, 1)$, $u_2 = (-1, 1, 4)$, $u_3 = (3, 5, 2)$.

Solution. Let $v = xu_1 + yu_2 + zu_3$. We have

$$\begin{bmatrix} 5 & -1 & 3 & 0 \\ 1 & 1 & 5 & -2 \\ 1 & 4 & 2 & 7 \end{bmatrix} \xrightarrow{R_1 \longleftrightarrow R_2} \begin{bmatrix} 1 & 1 & 5 & -2 \\ 5 & -1 & 3 & 0 \\ 1 & 4 & 2 & 7 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - 5R_1} \begin{bmatrix} 1 & 1 & 5 & -2 \\ 0 & -6 & -22 & 10 \\ 0 & 3 & -3 & 9 \end{bmatrix}$$

$$\xrightarrow{(-1/2)R_2} \begin{bmatrix} 1 & 1 & 5 & -2 \\ 0 & 3 & 11 & -5 \\ 0 & 3 & -3 & 9 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 1 & 5 & -2 \\ 0 & 3 & 11 & -5 \\ 0 & 0 & -14 & 14 \end{bmatrix} \xrightarrow{(-1/14)R_3} \begin{bmatrix} 1 & 1 & 5 & -2 \\ 0 & 3 & 11 & -5 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

We obtain $z = -1$, $3y - 11 = -5$ so $y = 2$, $x + 2 - 5 = -2$ so $x = 1$.

Hence $v = u_1 + 2u_2 - u_3$.

8. For each of the following subsets of $\mathbb{R}^4$, determine whether or not it is a subspace of $\mathbb{R}^4$. (Justify your answers.)

(a) W_1 is the set of all vectors (x_1, x_2, x_3, x_4) such that $x_1^2 + x_2^2 = x_4$.
Solution. Let $u = v = (1, 0, 0, 1)$. Then $u, v \in W_1$ but $u + v = (2, 0, 0, 2) \notin W_1$. So W_1 is not a subspace of $\mathbb{R}^4$.

(b) W_2 is the set of all vectors (x_1, x_2, x_3, x_4) such that $x_1 + x_4 = 3x_2 - 2x_3$.
Solution. W_2 is the solution set of the homogeneous system (in one equation):

$$x_1 - 3x_2 + 2x_3 - x_4 = 0.$$

Hence W_2 is a subspace of $\mathbb{R}^4$.